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Composition of Clans for Solving Linear Systems on Parallel Architectures

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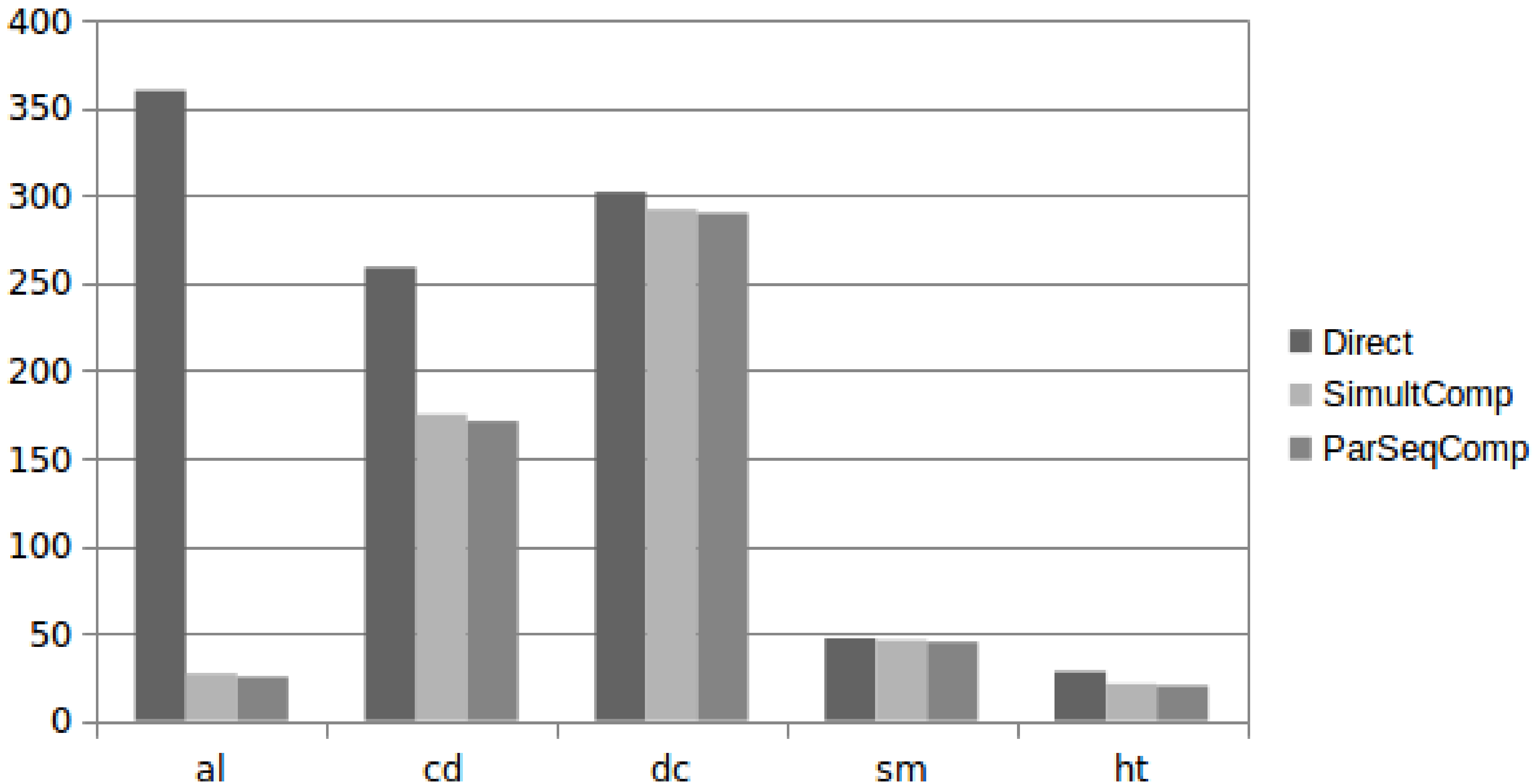
Speed-Up Solving Systems on 16 Nodes



Key Features

- Speed-up solving (especially Diophantine) systems of linear algebraic equations
- Sparse systems of specific form, namely "well decomposable into clans"
- Concept of a sign forms clans of equations
- Applicable to other algebraic structures with sign

Form of Obtained Matrix

$$A = \left\|\begin{matrix} A^{0,1} & \widehat{A}^{1} & 0 & 0 & 0 \\ A^{0,2} & 0 & \widehat{A}^{2} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{0,k} & 0 & 0 & 0 & \widehat{A}^{k} \end{matrix}\right\|$$

Divide and Sway

- **Decompose a given system into its clans**
- **Solve a system for each clan**
- **Solve a system of clans composition**
- **Or collapse the decomposition graph solving a system for each contracted edge**
- **Obtain a result in feasible time**

Algebraic structure

Numbers	Structure	Methods	Complexity
Complex	field	а) reduction: LU, QR; б) iteration methods	$O(n^3)$
Real			
Integer	ring	Normal forms: Hermite, Smith	$O(n^4)$
Nonnegative integer	monoid	Methods of Toudic (Silva) and Contejean	$O(2^n)$

Real-life matrices (systems)

- Matrix Market - https://math.nist.gov/MatrixMarket
- The SuiteSparse Matrix Collection https://sparse.tamu.edu
- Model Checking Contest – Petri net models https://mcc.lip6.fr

Basic software

Structure \ Type	Dense	Sparse
Field	LAPACK	UMFPACK
Ring	4ti2	ParAd+4ti2
Monoid	4ti2	ParAd

A Clan – Transitive Closure of Nearness Relation

$$C1: \begin{cases} -x_2 + x_3 - x_{15} \underline{+ x_{18}} = 0 \\ -x_2 + x_4 - x_{14} \underline{+ x_{18}} = 0 \\ -x_5 + x_6 - x_{16} \underline{+ x_{18}} = 0 \\ -x_{11} + x_{12} - x_{15} \underline{+ x_{18}} = 0 \\ -x_{13} + x_8 \underline{+ x_{18}} = 0 \end{cases}$$

Two equations are *near* if they contain the same variable having nonzero coefficients of the same sign

Decomposition into Clans

$$
\begin{cases}
-x_1 + x_2 - x_{18} = 0 \\
-x_2 + x_3 - x_{15} + x_{18} = 0 \\
-x_2 + x_4 - x_{14} + x_{18} = 0 \\
-x_4 + x_5 + x_{14} - x_{18} = 0 \\
-x_3 + x_5 + x_{15} - x_{18} = 0 \\
-x_5 + x_6 - x_{16} + x_{18} = 0 \\
-x_6 + x_7 + x_{16} - x_{19} = 0 \\
x_1 - x_7 + x_{19} = 0 \\
-x_8 + x_9 - x_{19} = 0 \\
-x_9 + x_{10} - x_{17} + x_{19} = 0 \\
-x_{10} + x_{11} + x_{17} - x_{18} = 0 \\
-x_{11} + x_{12} - x_{15} + x_{18} = 0 \\
-x_{12} + x_{13} + x_{15} - x_{18} = 0 \\
-x_{13} + x_8 + x_{18} = 0
\end{cases}
$$

$$
\text{C1: }
\begin{cases}
-x_2 + x_3 - x_{15} + x_{18} = 0 \\
-x_2 + x_4 - x_{14} + x_{18} = 0 \\
-x_5 + x_6 - x_{16} + x_{18} = 0 \\
-x_{11} + x_{12} - x_{15} + x_{18} = 0 \\
-x_{13} + x_8 + x_{18} = 0
\end{cases}
$$

$$
\text{C2: }
\begin{cases}
-x_1 + x_2 - x_{18} = 0 \\
-x_4 + x_5 + x_{14} - x_{18} = 0 \\
-x_3 + x_5 + x_{15} - x_{18} = 0 \\
-x_{10} + x_{11} + x_{17} - x_{18} = 0 \\
-x_{12} + x_{13} + x_{15} - x_{18} = 0
\end{cases}
$$

$$
\text{C3: }
\begin{cases}
x_1 - x_7 + x_{19} = 0 \\
-x_9 + x_{10} - x_{17} + x_{19} = 0
\end{cases}
$$

$$
\text{C4: }
\begin{cases}
-x_6 + x_7 + x_{16} - x_{19} = 0 \\
-x_8 + x_9 - x_{19} = 0
\end{cases}
$$

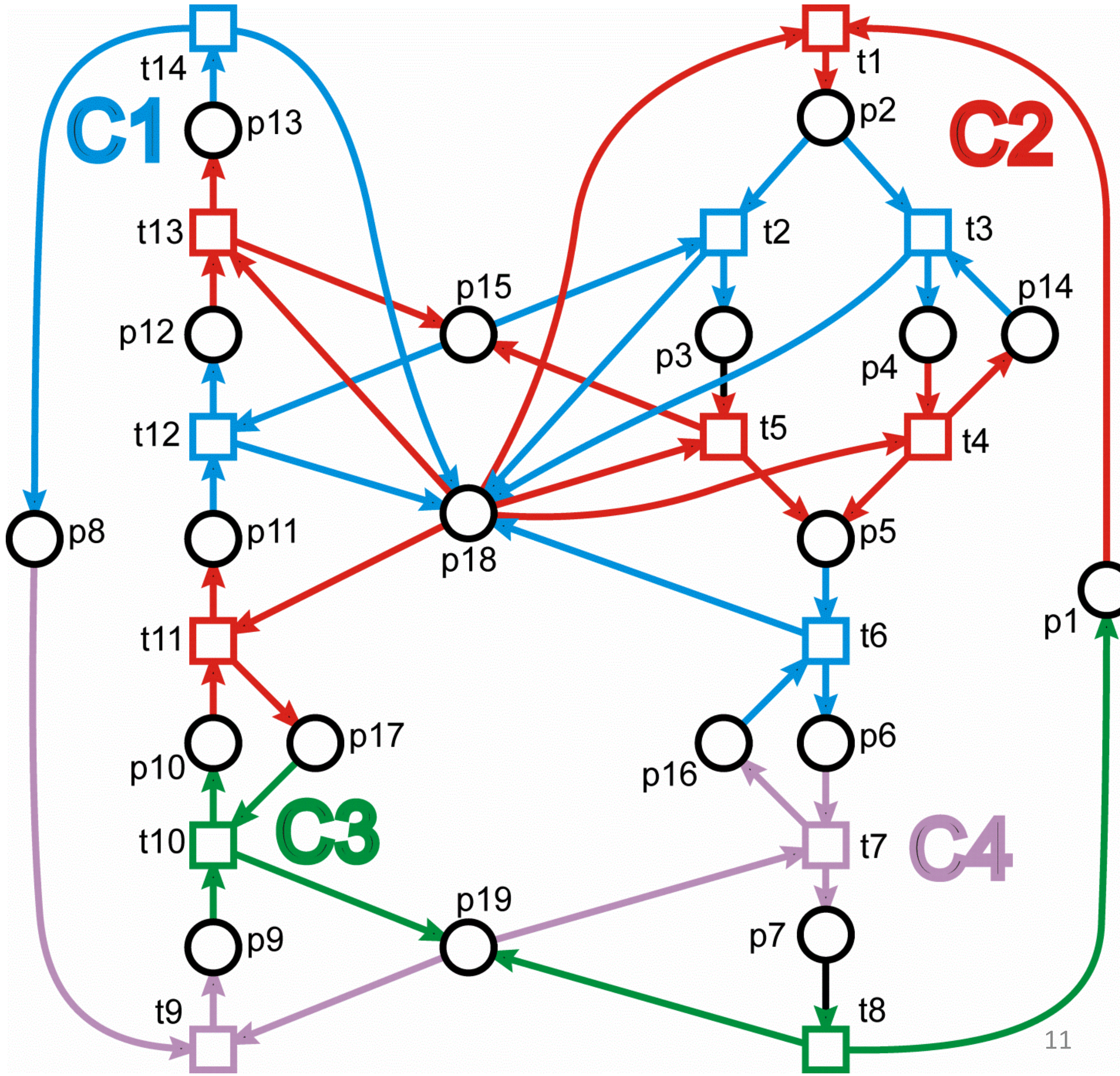
Systems and Directed Bipartite Graphs

Equation – transition (rectangle)

Variable – place (circle)

Positive sign – incoming arc of a place

Negative sign – outgoing arc of a place



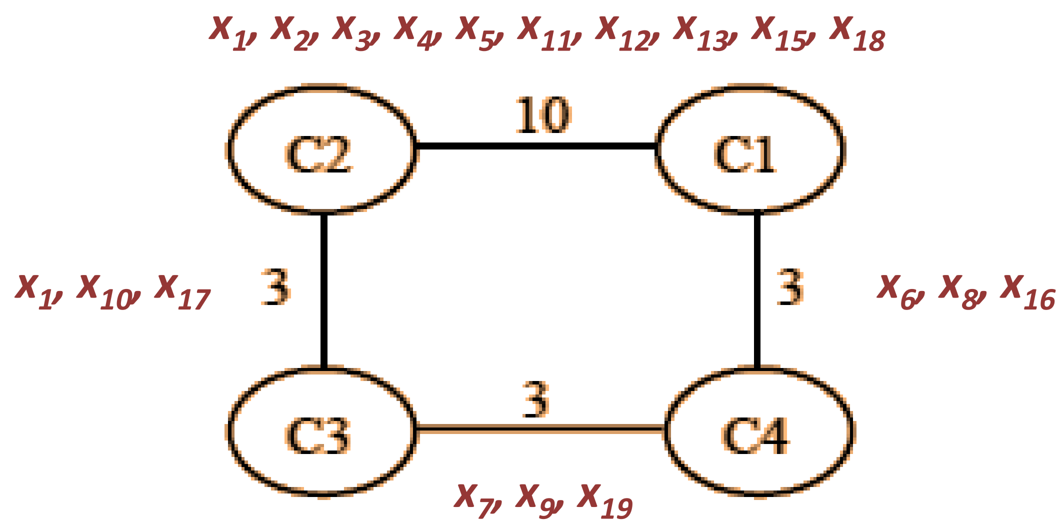
Decomposition Graph

$$
\text{C1:}\begin{cases}
-x_2+x_3-x_{15}+x_{18}=0\\
-x_2+x_4-x_{14}+x_{18}=0\\
-x_5+x_6-x_{16}+x_{18}=0\\
-x_{11}+x_{12}-x_{15}+x_{18}=0\\
-x_{13}+x_8+x_{18}=0
\end{cases}
$$

$$
\text{C2:}\begin{cases}
-x_1+x_2-x_{18}=0\\
-x_4+x_5+x_{14}-x_{18}=0\\
-x_3+x_5+x_{15}-x_{18}=0\\
-x_{10}+x_{11}+x_{17}-x_{18}=0\\
-x_{12}+x_{13}+x_{15}-x_{18}=0
\end{cases}
$$

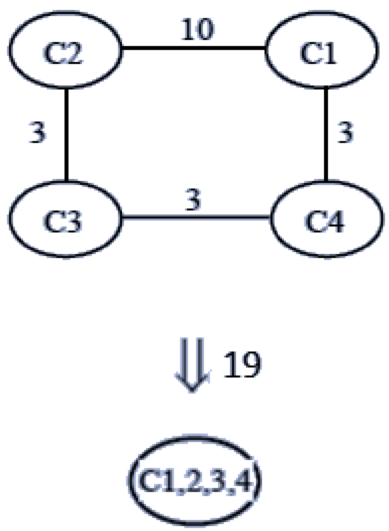
$$
\text{C3:}\begin{cases}
x_1-x_7+x_{19}=0\\
-x_9+x_{10}-x_{17}+x_{19}=0
\end{cases}
$$

$$
\text{C4:}\begin{cases}
-x_6+x_7+x_{16}-x_{19}=0\\
-x_8+x_9-x_{19}=0
\end{cases}
$$

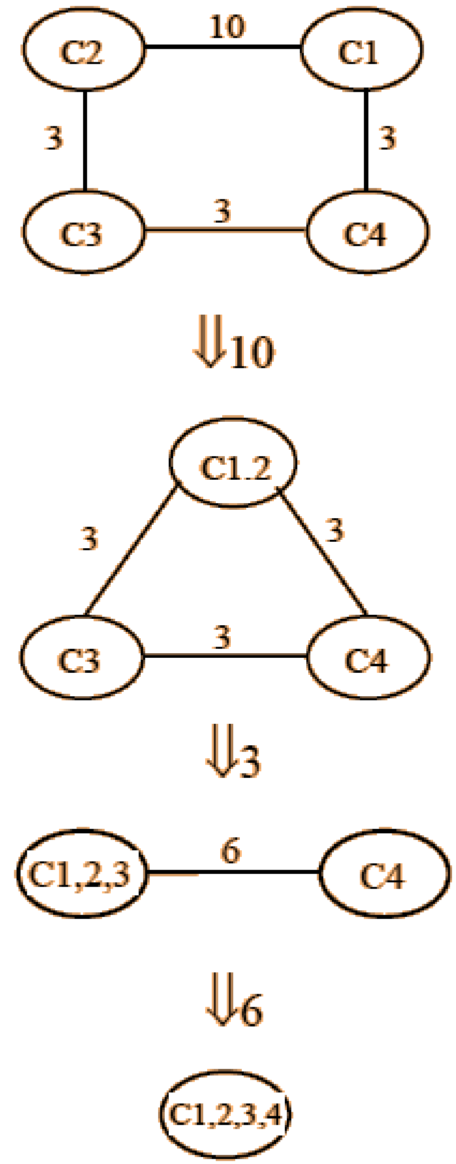


Collapse of Decomposition Graph

I.



II.



Decomposition into Clans as Matrix Reordering

- Clan – subset of equations
- Decomposition into clans – reordering of rows
- Linear complexity in the number of nonzero elements
- Classification of variables into contact and internal (on clans) reorders columns
- Combination of a block-column and a block diagonal matrices

Systems of Equations (Inequalities)

$$A \cdot \bar{x} = \bar{b}$$

its general solution

$$\bar{x} = \bar{x}' + G \cdot \bar{y}$$

Consider a system as a predicate

$$S(\bar{x}) = L_1(\bar{x}) \wedge L_2(\bar{x}) \wedge \ldots \wedge L_m(\bar{x})$$

$$L_i(\bar{x}) = (\bar{a}^i \cdot \bar{x} = 0), \qquad \mathfrak{I} = \{L_i\}$$

Relations on the Set of Equations

Relation of nearness: $L_i \circ L_j,$

$$\exists x_k \in X: \quad a_{i,k}, a_{j,k} \neq 0, \; sign(a_{i,k}) = sign(a_{j,k})$$

Statement. The relation of nearness is reflexive and symmetric.

Relation of clan: $L_i \bigcirc L_j$

$$L_{l_1}, L_{l_2}, \ldots, L_{l_k}: \qquad L_i \circ L_{l_1} \circ \ldots \circ L_{l_k} \circ L_j$$

Theorem. The relation of clan is an equivalence relation (reflexive, symmetric, and transitive).

Corollary. Relation of clan defines *a partition* of the set of equations; an element of this partition is called *a clan*.

Classification of Variables

Variables of a clan: X^j

$$X^j = X(C^j) = \{x_i \mid x_i \in X, \exists L_k \in C^j : a_{k,i} \neq 0\}$$

Internal variables of a clan: $\hat{X}^j$

$$x_i \in X(C^j), \quad \forall C^l, l \neq j: \; x_i \notin X^l$$

Contact variables: X^0

$$\exists C^j, C^l: \quad x_i \in X^j, \quad x_i \in X^l$$

Contact variables of a clan: $\breve{X}^j$

$$X^j = \hat{X}^j \cup \breve{X}^j, \quad \hat{X}^j \cap \breve{X}^j = \varnothing$$

Theorem. A contact variable belongs to two clans exactly entering one clan with sign plus and the other clan with sign minus.

Decomposition of System Matrix

Clans/variables	X^0	$\widehat{X}^1$	$\widehat{X}^2$	$\ldots$	$\widehat{X}^k$
C^1	$A^{0,1}$	$\widetilde{A}^1$	0	$\ldots$	0
C^2	$A^{0,2}$	0	$\widetilde{A}^2$	$\ldots$	0
. . .	. . .	. . .	. . .	. . .	. . .
C^k	$A^{0,k}$	0	0	$\ldots$	$\widetilde{A}^k$

Composition of Clans

1. Solve the system separately for each clan: $\bar{x}^j = G^j \cdot \bar{y}^j$

$$A^j \cdot \bar{x}^j = 0, \quad A^j = \left\| \breve{A}^j \quad \widehat{A}^j \right\|, \quad \bar{x}^j = \left\| \begin{matrix} \breve{\bar{x}}^j \\ \widehat{\bar{x}}^j \end{matrix} \right\|$$

2. Solve a system of composition of clans for contact variables:

$$G_i^j \cdot \bar{y}^j = G_i^l \cdot \bar{y}^l$$ or $$F \cdot \bar{y} = 0 : \qquad \bar{y} = R \cdot \bar{z}$$

3. Recover sought solutions:

$$\bar{x} = G \cdot \bar{y}, \quad G = \left\| \begin{matrix} J^1 & \widehat{G}^1 & 0 & 0 & 0 \\ J^2 & 0 & \widehat{G}^2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ J^k & 0 & 0 & 0 & \widehat{G}^k \end{matrix} \right\|^T, \qquad \bar{x} = G \cdot R \cdot \bar{z},$$

General Solutions Obtained via Composition of Clans

Theorem 1. A general solution of homogeneous system is:

$$\bar{x} = H \cdot \bar{z}, \quad H = G \cdot R$$

Theorem 2. A general solution of heterogeneous system is:

$$\bar{x} = \bar{y}'' + H \cdot \bar{z}, \quad \bar{y}'' = \bar{x}' + G \cdot \bar{y}', \quad H = G \cdot R$$

Statement. Speed-up of computations is about: $\dfrac{M(q)}{k \cdot nz + k \cdot M(p)}$

For exponential methods – exponential speed-up: $O(2^{q-p})$

Example: Decomposition into Clans

$$A = \begin{Vmatrix} 1 & 0 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

$$C^1 = \{L_1, L_2, L_5, L_6\} \qquad C^2 = \{L_3, L_4, L_7, L_8, L_9\}$$

$$X^1 = \{x_3, x_6, x_8, x_{10}, x_1, x_2, x_7\} \qquad X^2 = \{x_3, x_6, x_8, x_{10}, x_4, x_5, x_9\}$$

$$\widehat{X}^1 = \{x_1, x_2, x_7\} \qquad \widehat{X}^2 = \{x_4, x_5, x_9\}$$

$$X^0 = \breve{X}^1 = \breve{X}^2 = \{x_3, x_6, x_8, x_{10}\}$$

Example: Renumeration of Equations and Variables

$$nx = \begin{pmatrix} 3 & 6 & 8 & 10 & 1 & 2 & 7 & 4 & 5 & 9 \end{pmatrix}$$

$$nL = \begin{pmatrix} 1 & 2 & 5 & 6 & 3 & 4 & 7 & 8 & 9 \end{pmatrix}$$

$$A = \begin{Vmatrix} 2 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{Vmatrix}$$

Example: Solution of Systems for Clans

$$G^1 = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}^T, \qquad \bar{y}^1 = (y_1^1, y_2^1, y_3^1)^T$$

$$G^2 = \begin{Vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}^T, \qquad \bar{y}^2 = (y_1^2, y_2^2)^T$$

Example: Solution of System for Contact Variables

$$\begin{cases} y_1^1 - y_1^2 = 0, \\ y_1^1 - y_1^2 = 0, \\ y_2^1 - y_1^2 = 0, \\ y_2^1 - y_1^2 = 0. \end{cases}$$

$$R = \begin{Vmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}^T$$

Example: Composition of Source System Solution

$$G = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}^T \quad \text{or} \quad G = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}^T$$

$$H = R \cdot G = \begin{Vmatrix} 1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{Vmatrix}^T$$

Sequential Contraction of Graphs as a Scheme of Solving System

Graph of system decomposition into its clans: $G = (V, E, W)$

$V = \{v\}, \quad v \leftrightarrow C$ vertices correspond to clans

$E \subseteq V \times V$ edges connect clans having common contact variables

$$v_1 v_2 \in E \Leftrightarrow \exists x \in X^0 : (I(x) = C^1 \wedge O(x) = C^2) \vee (I(x) = C^2 \wedge O(x) = C^1)$$

$W : (V \rightarrow \mathrm{N}) \bigcup (E \rightarrow \mathrm{N})$ weight function;

$w(v)$ number of clan variables;

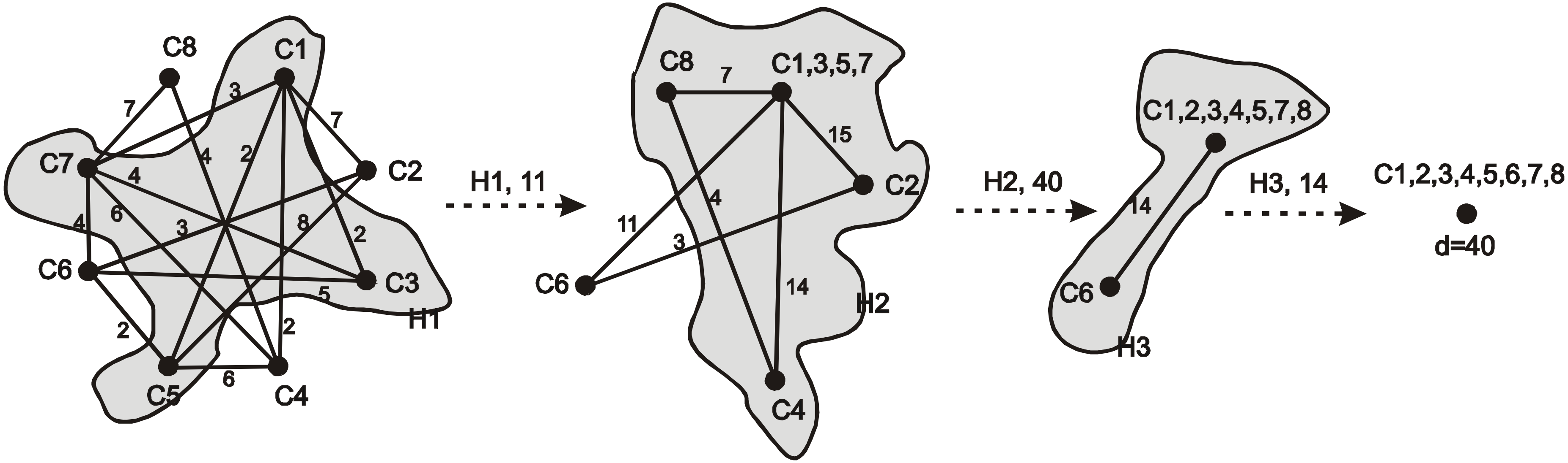
$w(v,u)$ number of contact variables;

$$w(v) \geq \sum_u w(v,u)$$

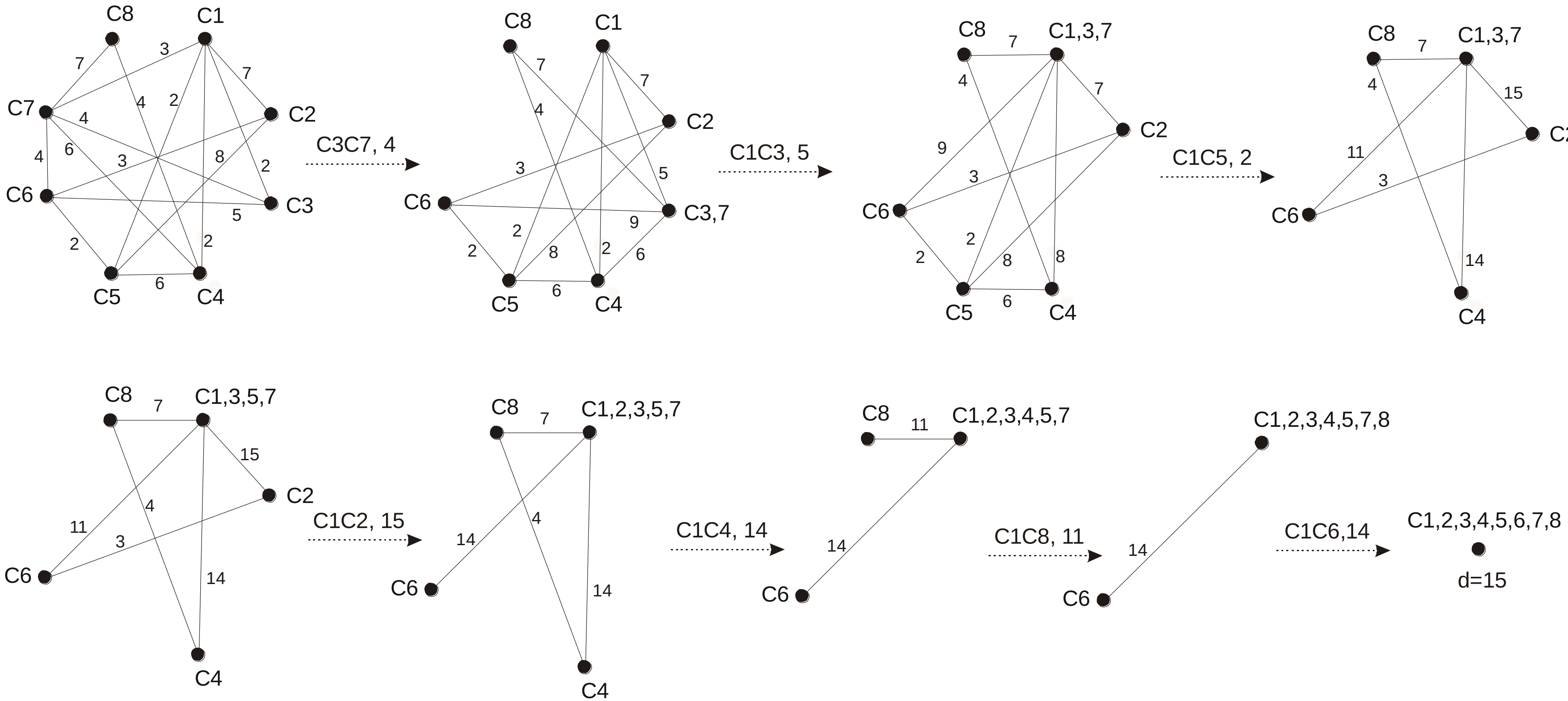
Collapse of graph:

$$G = G^0 \xrightarrow[d_1]{V^1} G^1 \xrightarrow[d_2]{V^2} G^2 \ldots \xrightarrow[d_k]{V^k} G^k$$

Collapse of Subgraphs

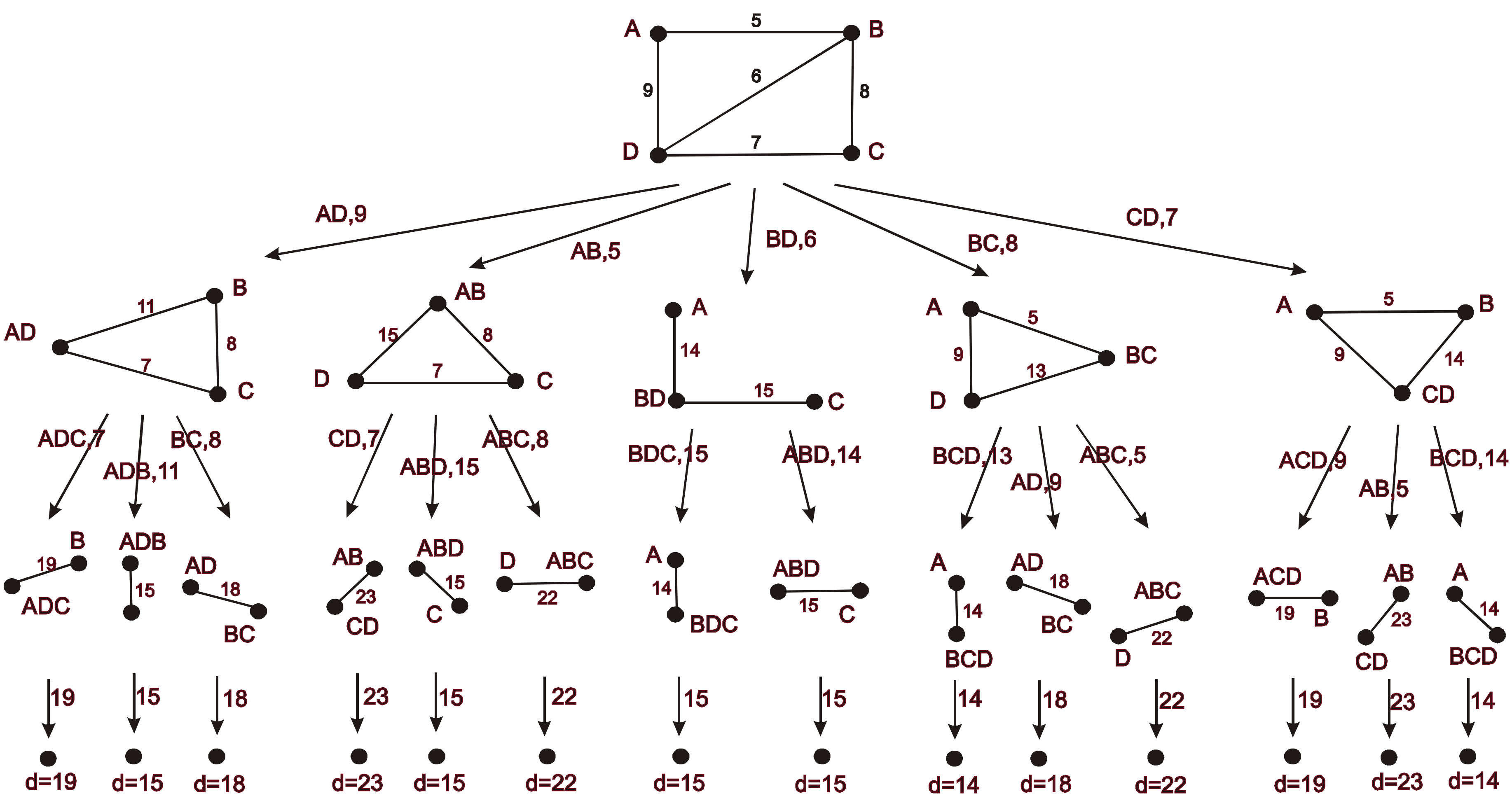


Edge collapse of graph



Collapse width 15 – dimension of systems.

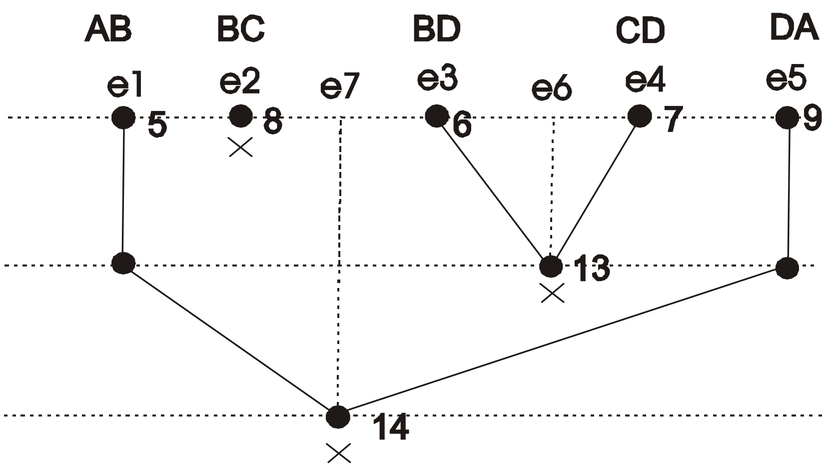
An Exhaustive Search of Edge Collapse



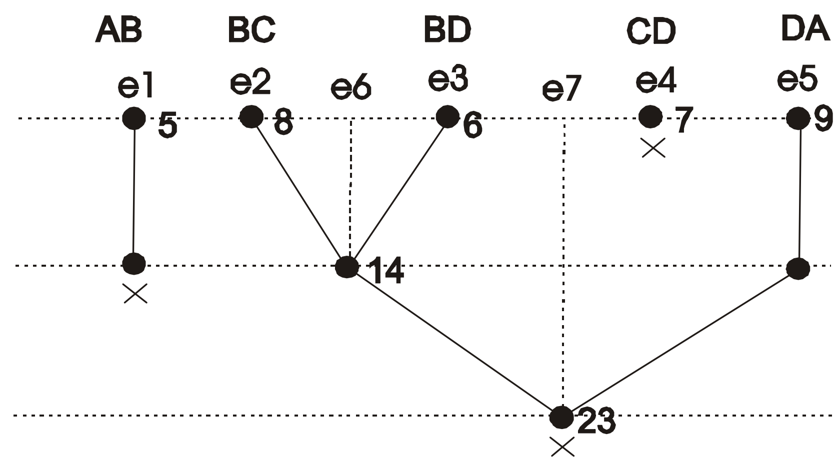
A Partial Lattice of Collapse

$$e_1^i \ll e_3^{i+1} \Leftrightarrow e_3^{i+1} = e_1^i \vee e_3^{i+1} = e_1^i + e_2^i$$

$p_i = p_{i-1} - 1 - t$ - number of edges t - number of triangles



а) BC, BCD, ABCD

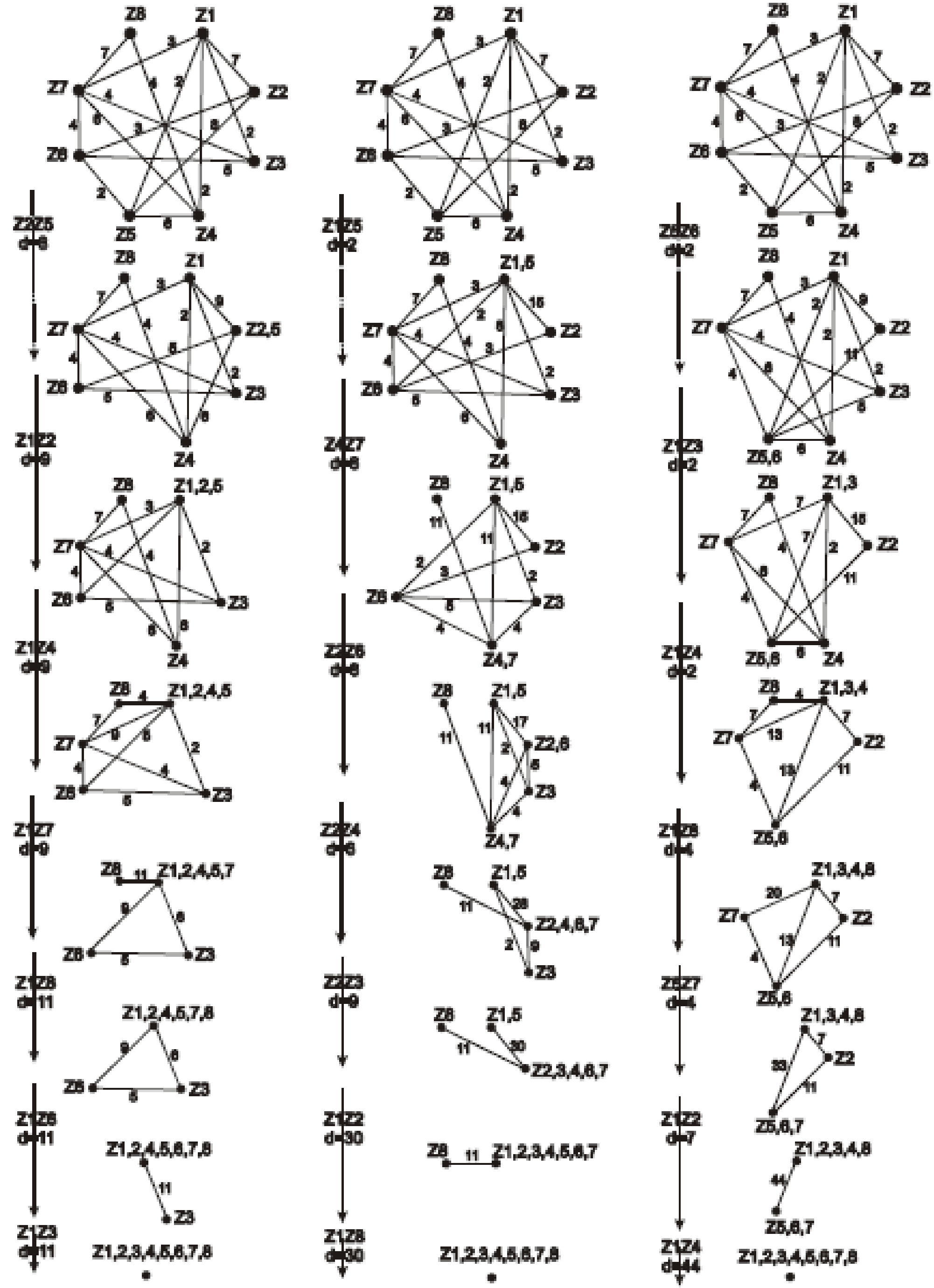


б) CD, AB, ABCC

Statement. Each edge on a step of a collapse is a sum of some edges of the source graph.

Comparing Heuristic Strategies of Edge Collapse

(maximal, random, and minimal edge)



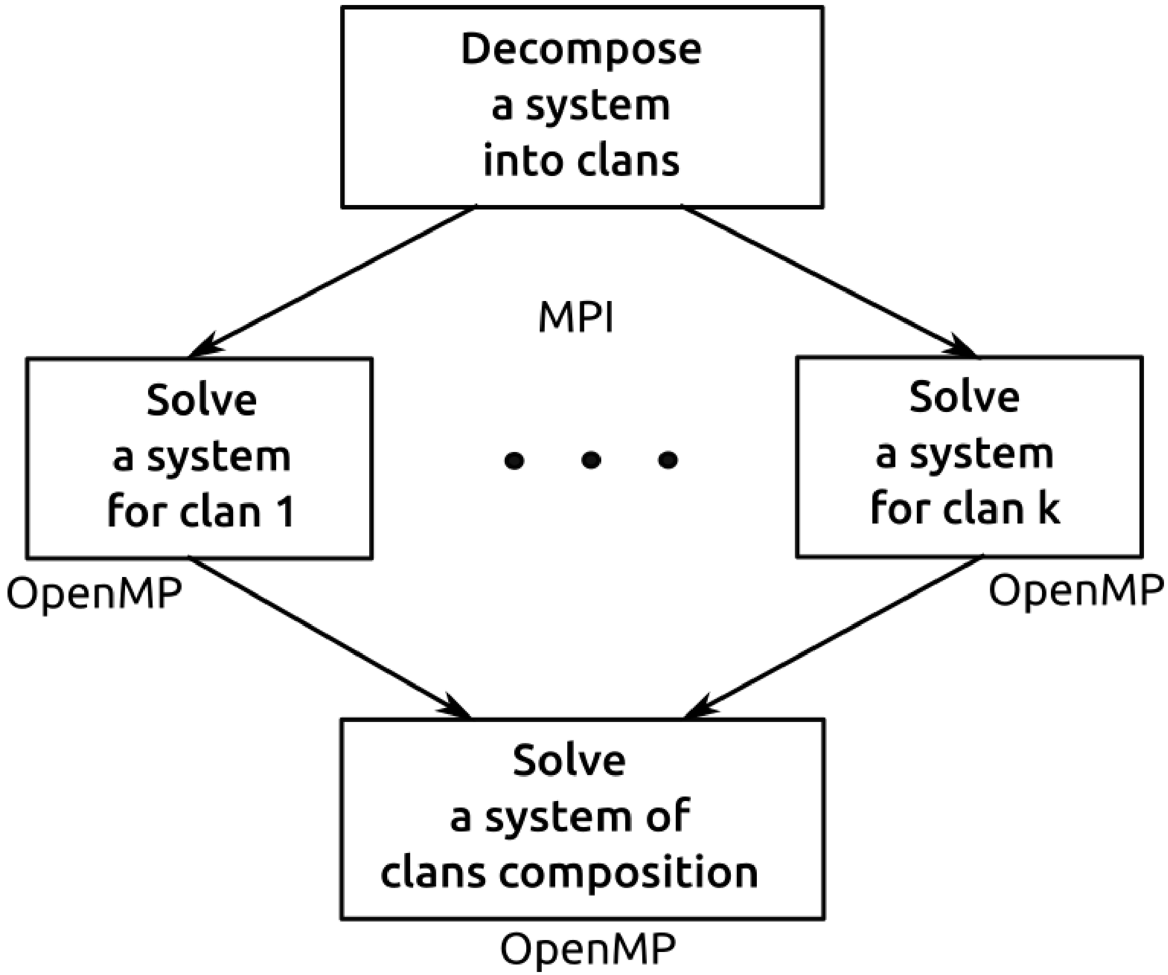
Comparison of Collapse Strategy for Random Graphs

Number of graph vertices	Denseness of graph (%)	Width of simultaneous collapse	Width of sequential collapse					
			Maximal edge		Random edge		Minimal edge	
			Width	%	Width	%	Width	%
20	20	442	35	7.9	191	44.6	231	52.3
	40	869	66	7.6	367	42.2	533	61.3
	60	1372	102	7.4	651	47.4	829	60.4
	80	1825	160	8.8	876	48.0	990	54.2
40	20	1836	73	4.0	632	34.4	1002	54.6
	40	3699	139	3.8	1664	45.0	2133	57.7
	60	5539	214	3.9	2665	48.1	2948	53.2
	80	7354	314	4.3	3608	49.0	3908	53.1
100	20	11602	160	1.4	4827	41.6	5829	50.2
	40	22973	316	1.4	7617	33.2	12341	53.7
	60	34334	501	1.5	13282	38.7	17559	51.1
	80	45582	754	1.7	17144	37.6	23008	50.5
200	20	46073	288	0.63	19673	42.7	23781	51.6
	40	91715	612	0.67	42260	46.0	91715	50.5
	60	137684	997	0.72	67609	49.1	68957	50.0
	80	183652	1486	0.81	91015	49.6	91669	49.9

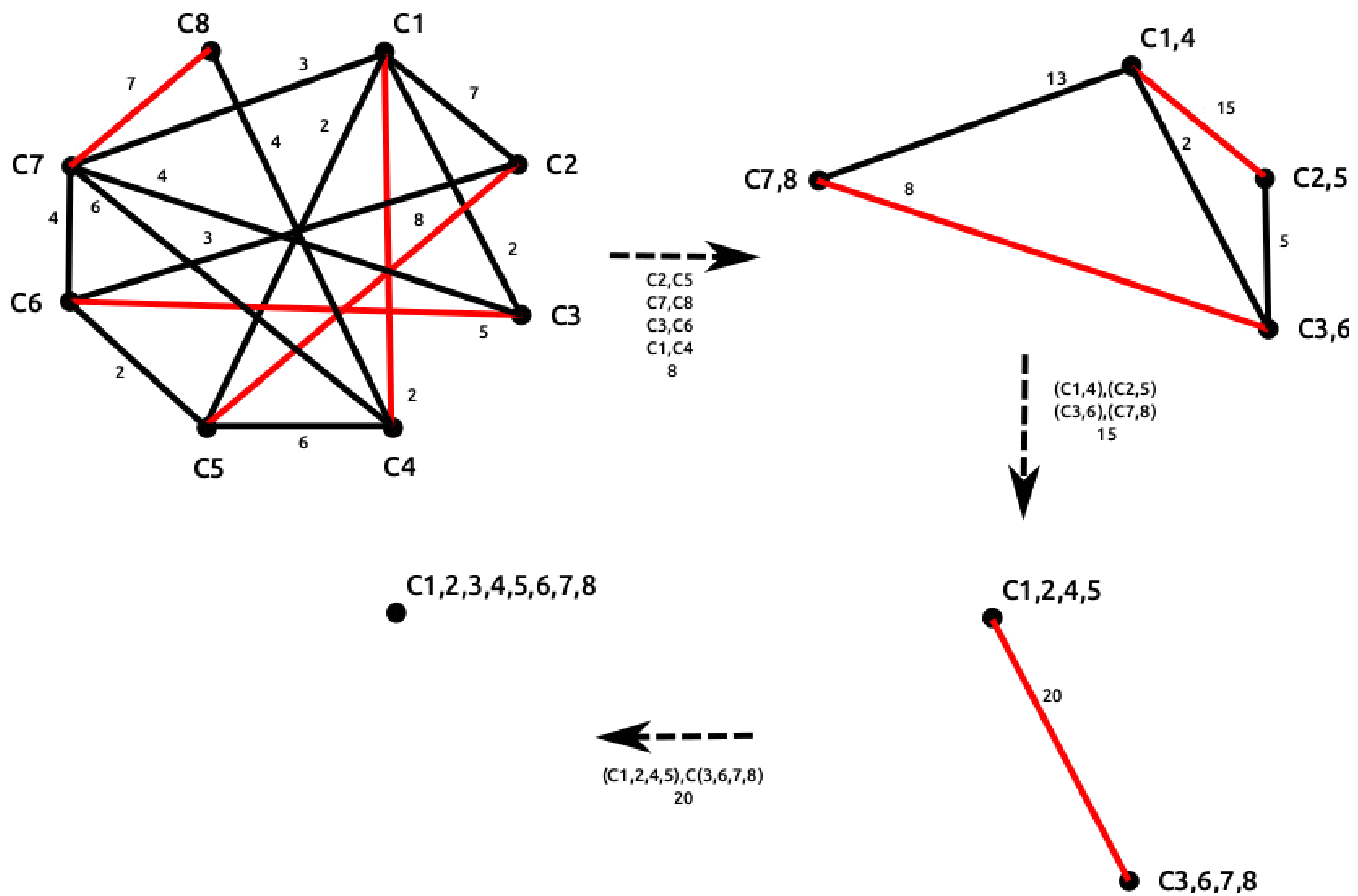
Software

- ***Deborah* – decomposition into clans, 2004**
- ***Adriana* – solving a homogenous system via (a) simultaneous or (b) sequential composition of clans, 2005**
- ***ParAd* – solving a homogenous system via (a) simultaneous or (b) parallel-sequential composition of clans on *parallel architectures*, 2017**

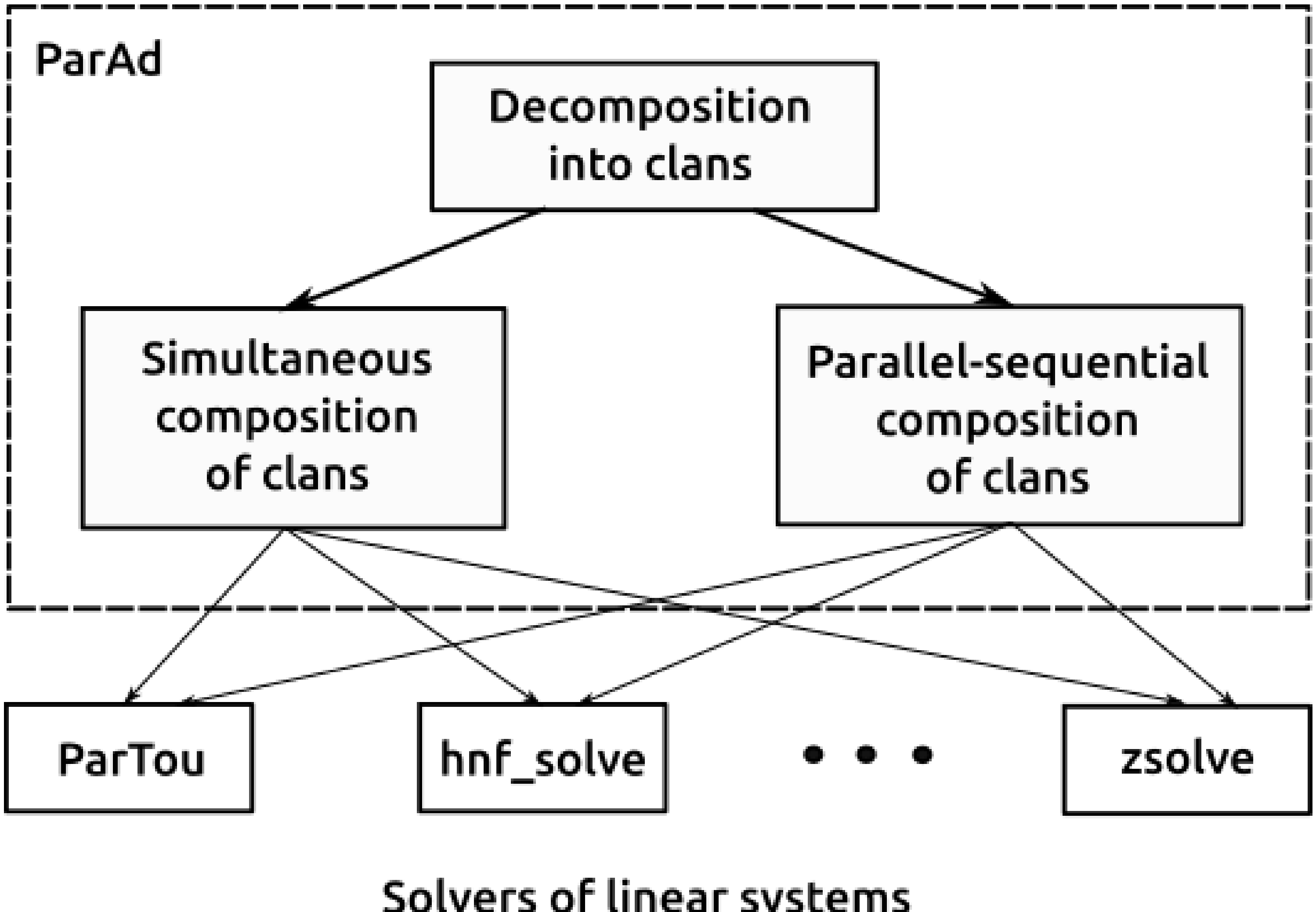
Composition of Clans on Parallel Architectures



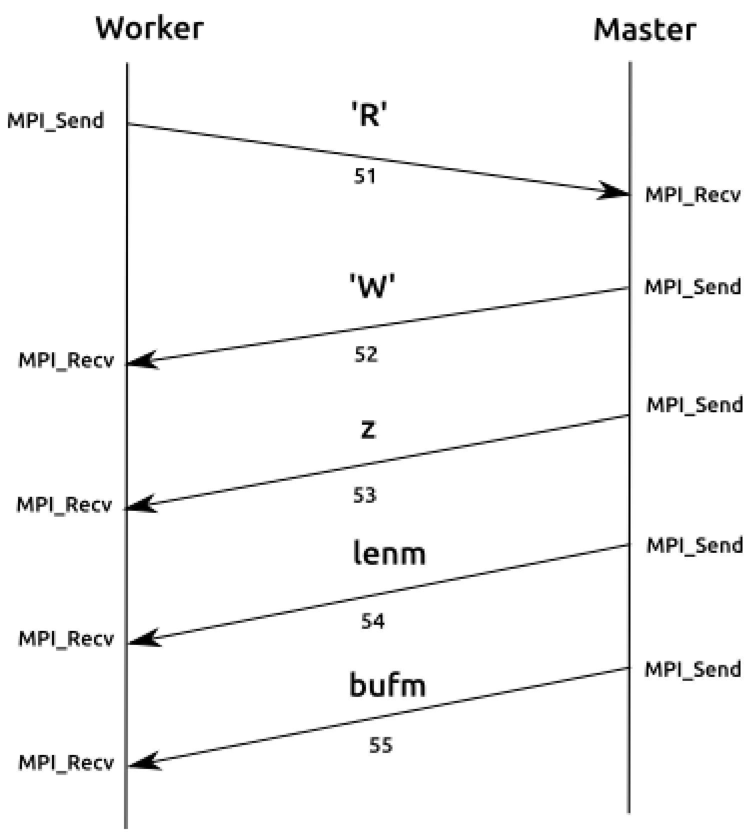
Parallel-sequential Composition of Clans



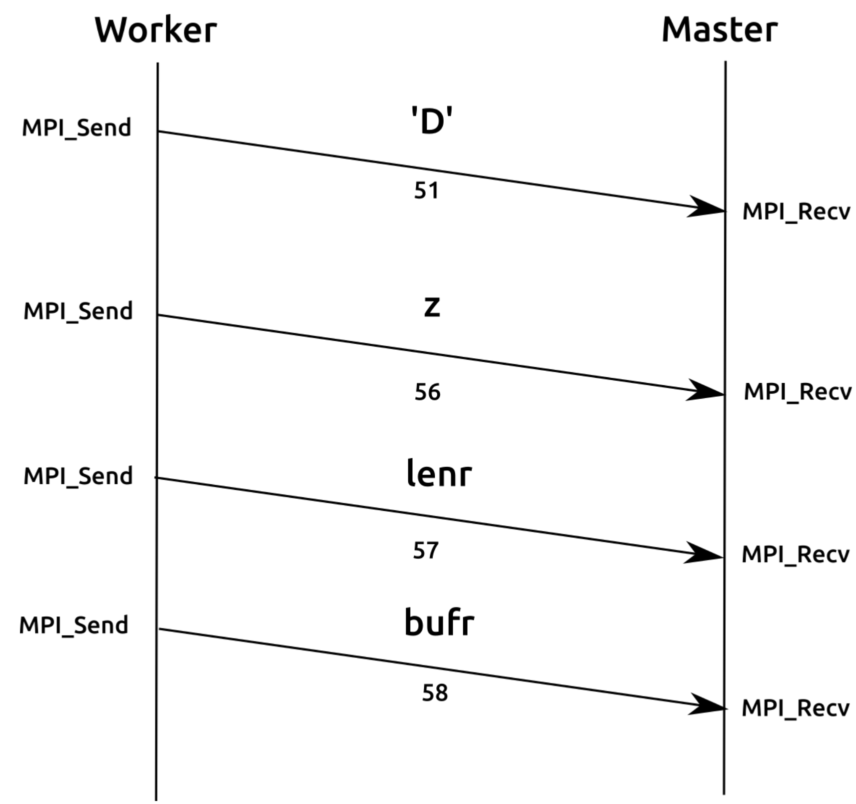
ParAd – Parallel Adriana



Protocols of Master-Worker Communication

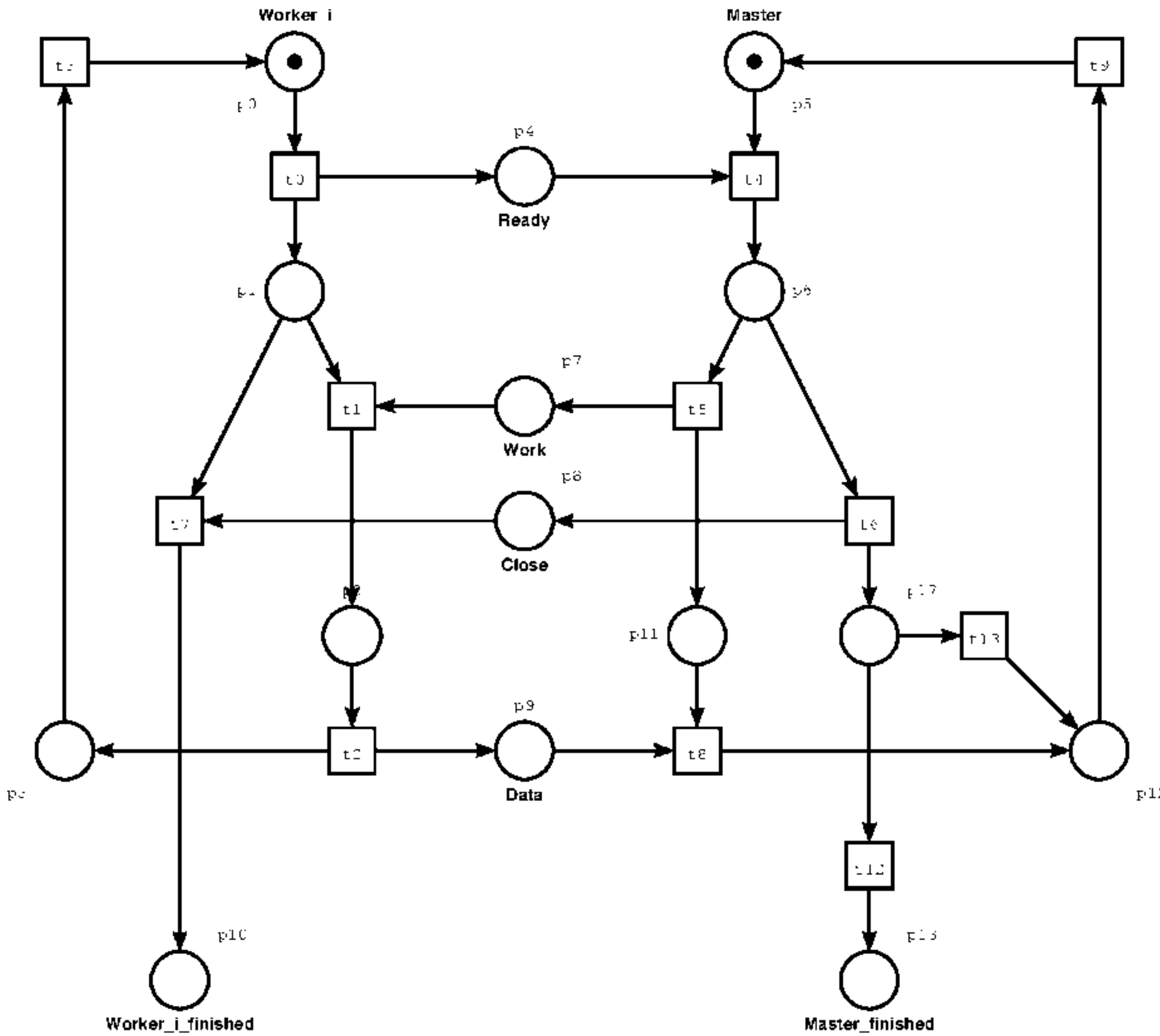


(a) Sending system

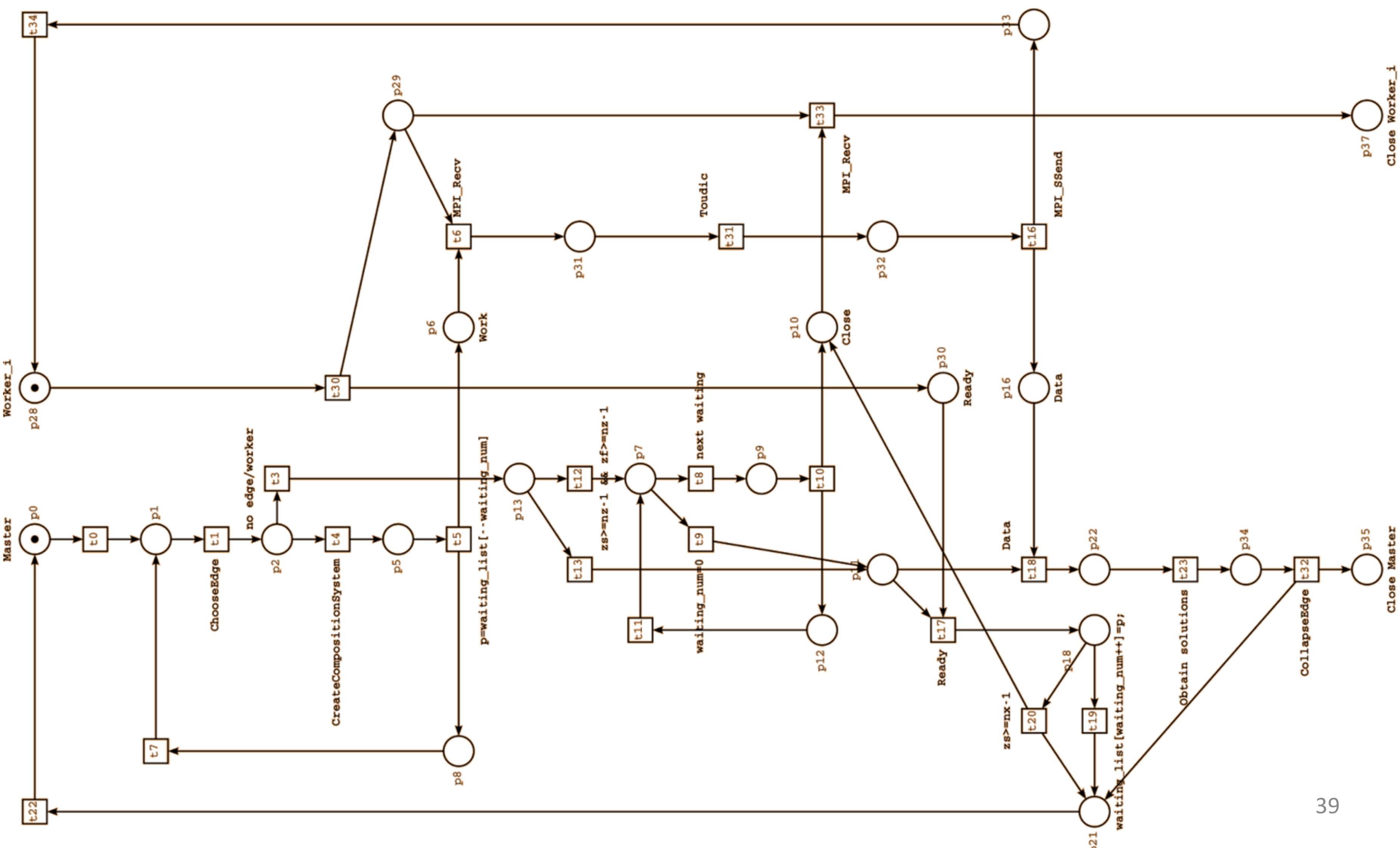


(b) Receiving solution

Master-Worker Basic Communication Model



Parallel-Sequential Composition Communication Model



Run ParAd

- Run with mpirun

```
>mpirun -n 5 ./ParAd -c -r zsolve tcp.spm tcp-pi.spm
>mpirun -n 10 ./ParAd -s -t -d 1 tcp.spm tcp-ti.spm
```

- Run with Slurm

```
>srun -N 10 ./ParAd -s -t -d 1 tcp.spm tcp-ti.spm
```

- SPM – simple sparse matrix format:

```
i j a[i][j]
```

- Check decomposability (Matrix Market Format)

```
>toclans lp_cre_d.mtx
```

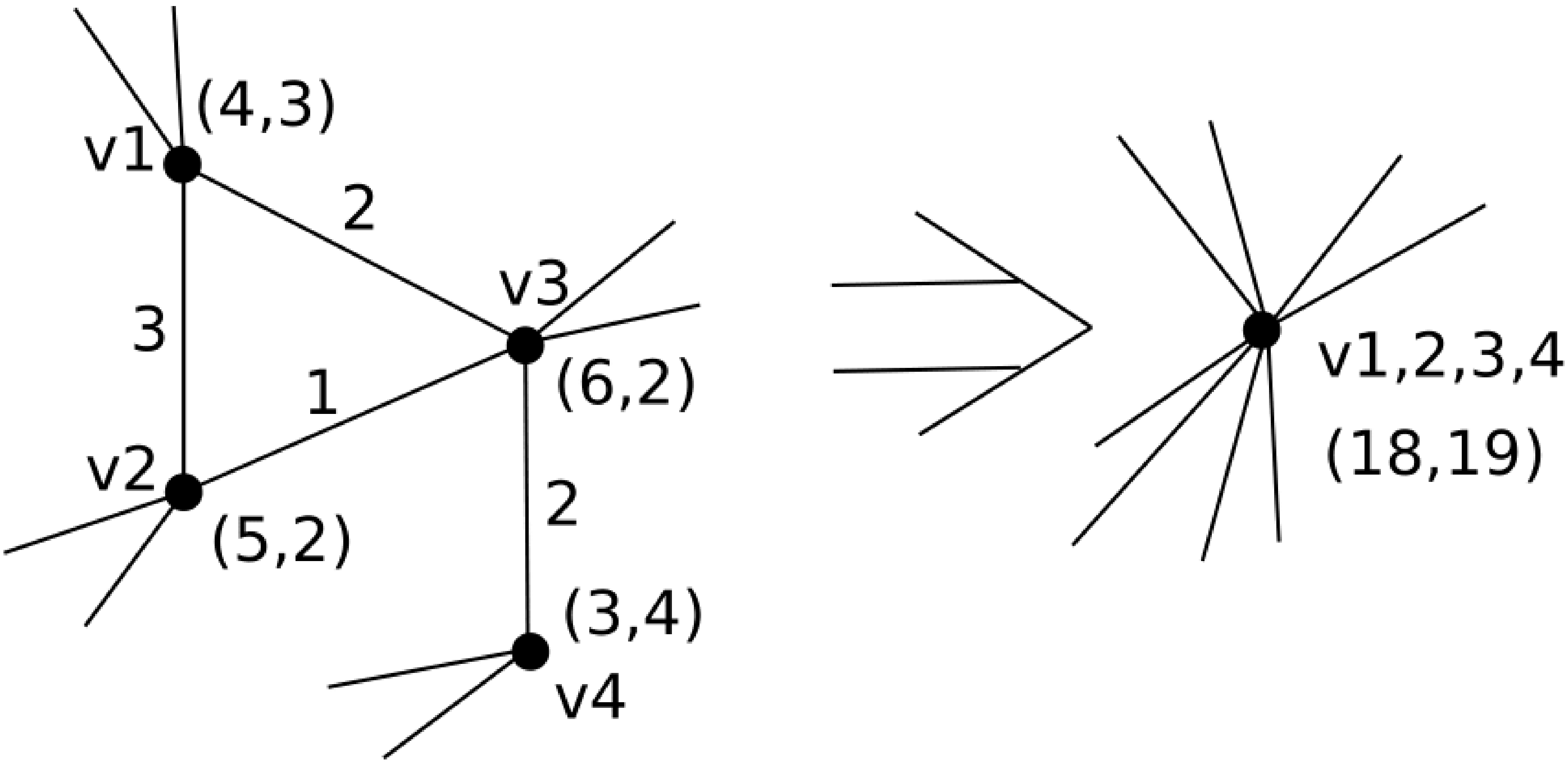
Aggregation of Clans for Workload Balancing

- A clan is a sum of minimal clans
- The maximal clan size restricts granulation
- Many small clans lead to heavy communication load
- Balancing: create clans having size close to the maximal
- Key: -a val
- Aggregation steeds-up about 20%

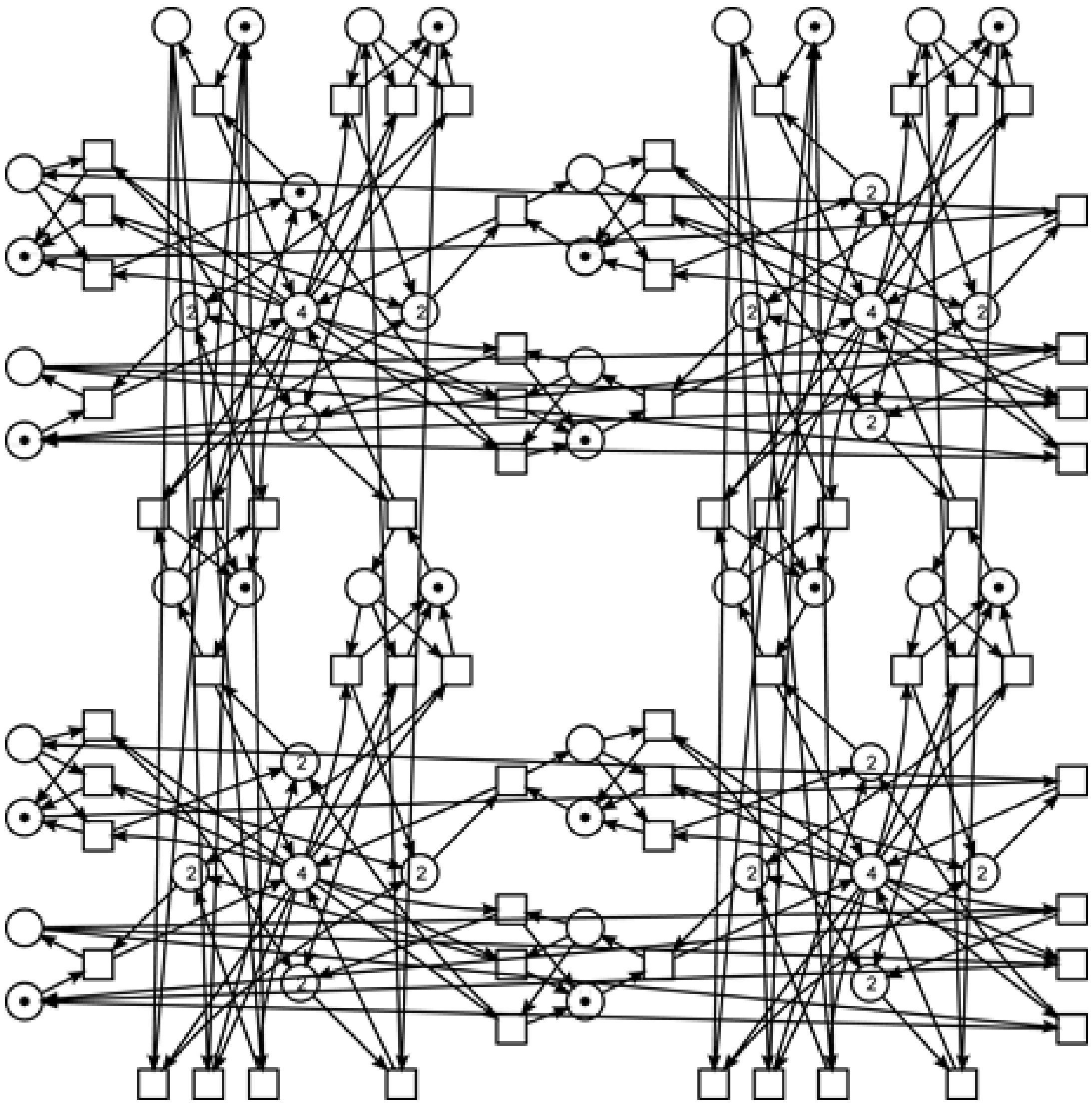
Load balancing

- Dynamic on demand – appoint a clan to a free node (version 1.1)
- Static – aggregate minimal clans into big clans according to the number of available computing nodes
- Hybrid – pre-aggregation to equal size and then dynamic scheduling clans to nodes (version 1.2)

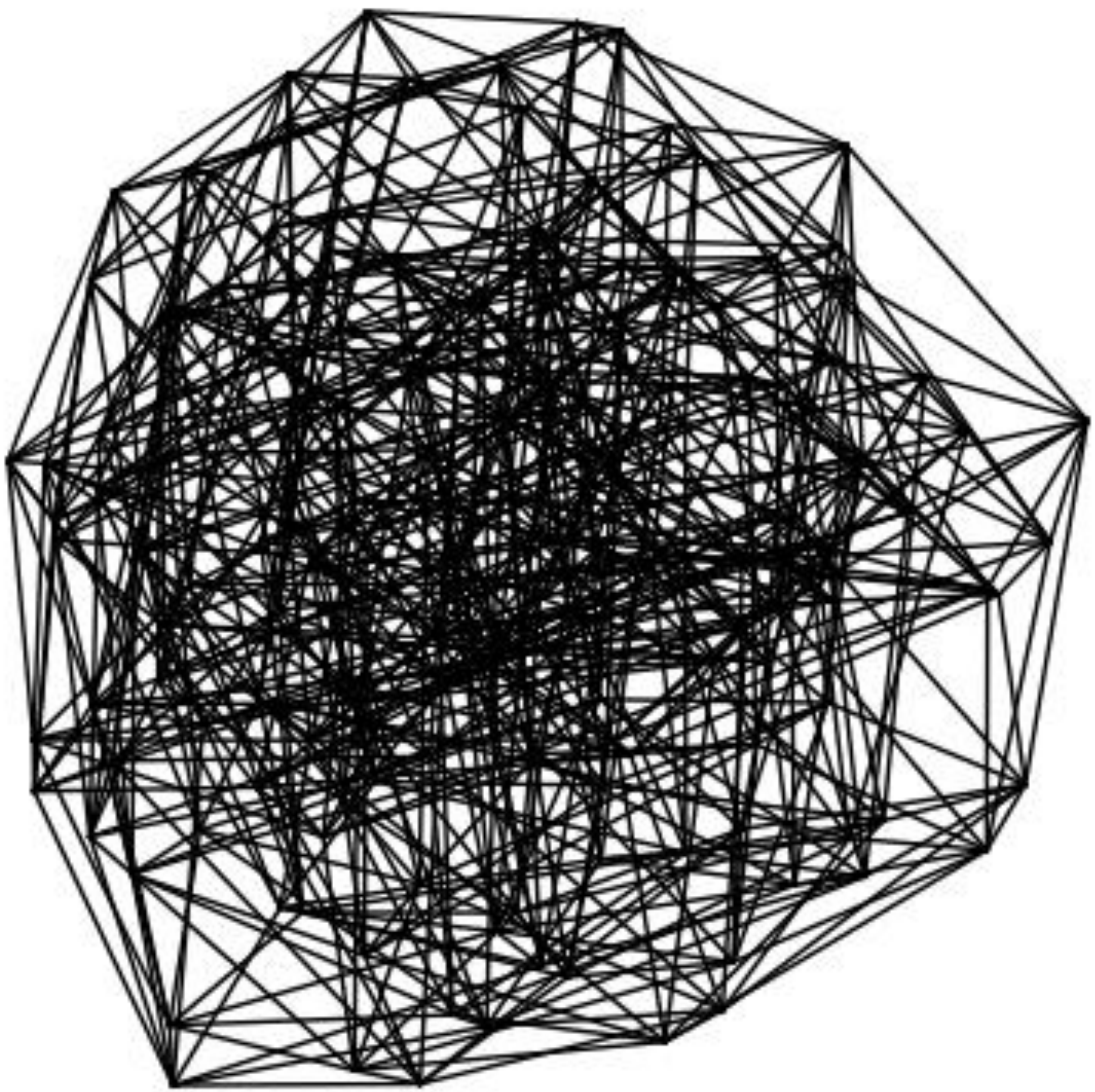
Aggregation idea



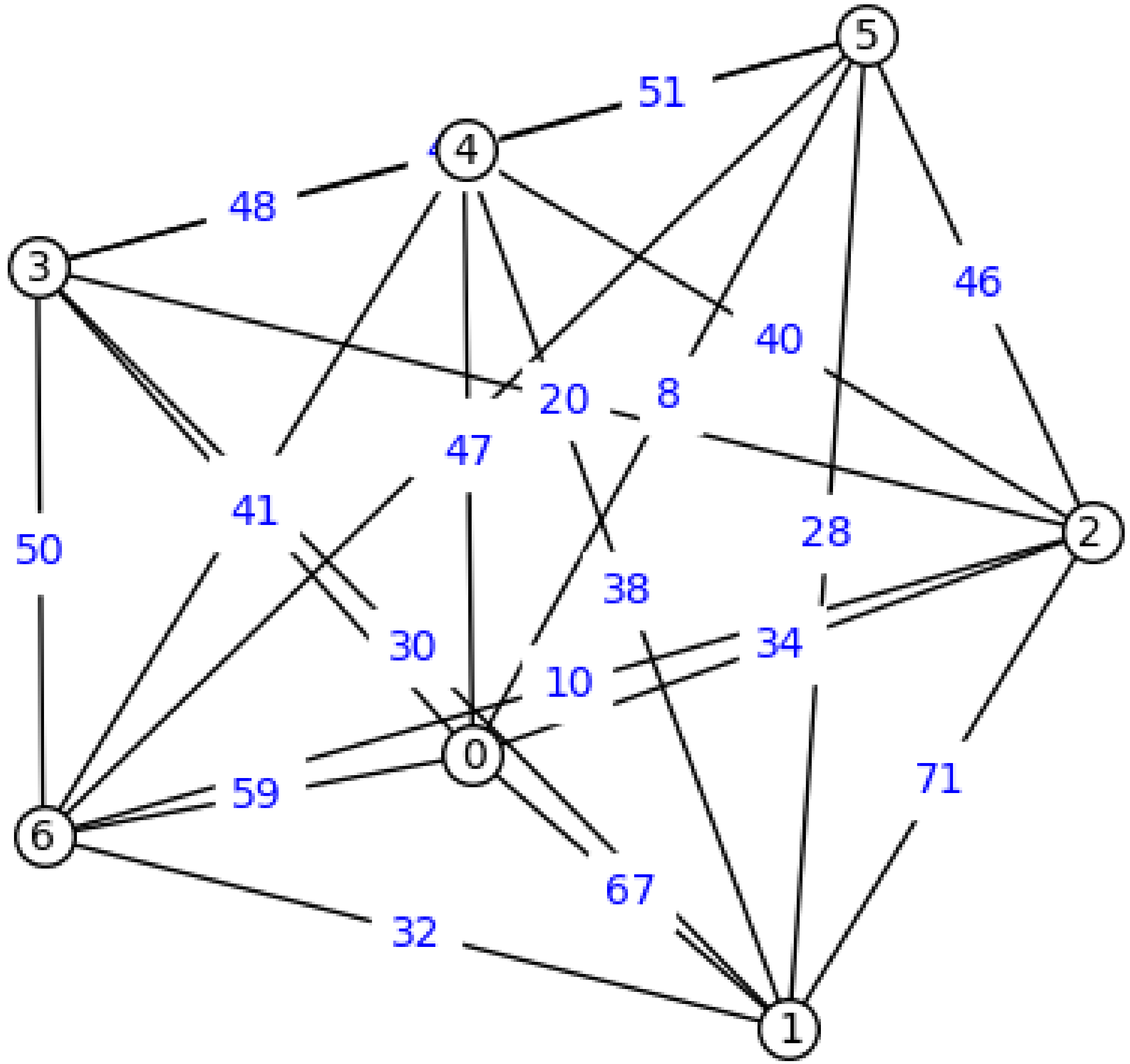
Hypertorus switching grid, 2D, 2x2 model



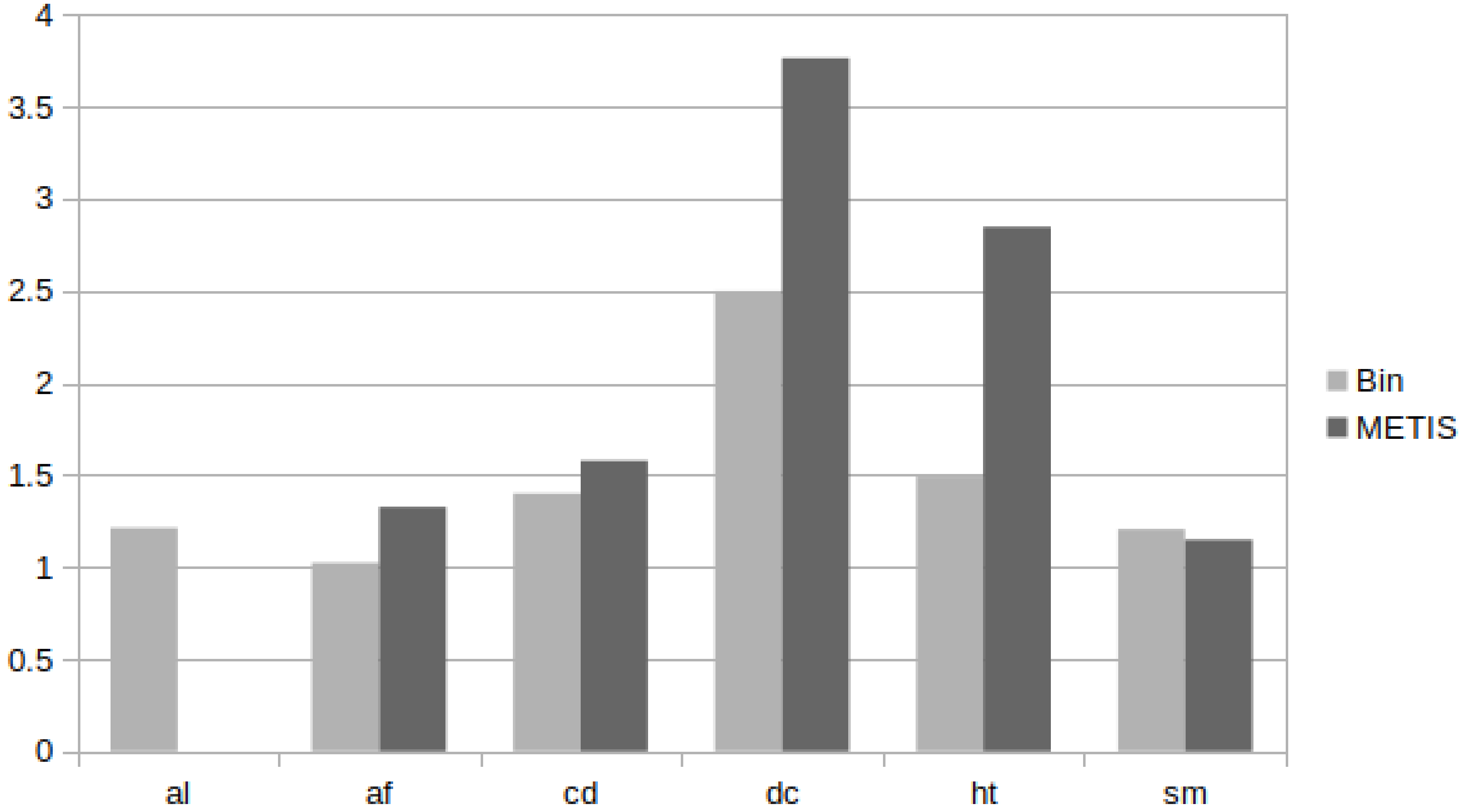
Decomposition of hypertorus 4D, 3x3



Aggregation by METIS into 7 clans



Extra speed-up because of aggregation



Solving Systems over Real Numbers

- How to solve a linear system for a non-square matrix (what software to use)?
- A variant: LAPACK, SVD
- A problem – accumulation of errors
- Preference to simultaneous composition
- Clans with SVD speed-up about 2 times on 16 nodes

Conclusions

- **Composition of clans speeds-up solving linear systems of equations**
- **Decomposition into clans is linear in the number of nonzero elements**
- **Many application area matrices are decomposable into clans**
- **Aggregation of clans provides load-balancing and gives additional speed-up**
- **The best total speed-up obtained is 180 times on 16 nodes with 20 cores each**

Basic references

- Zaitsev D.A., Tomov S., Dongarra J. Solving Linear Diophantine Systems on Parallel Architectures, IEEE Transactions on Parallel and Distributed Systems, 30(05), 2019, 1158-1169.
- Zaitsev D.A. Sequential composition of linear systems' clans, Information Sciences, Vol. 363, 2016, 292–307.
- Zaitsev D.A. Compositional analysis of Petri nets, Cybernetics and Systems Analysis, Volume 42, Number 1 (2006), 126-136.
- Zaitsev D.A. Decomposition of Petri Nets, Cybernetics and Systems Analysis, Volume 40, Number 5 (2004), 739-746.

Directions for future work

- A library that implements composition of clans independently from data types and solvers
- Multi-core implementation of decomposition
- Multi-core implementation of sparse matrix multiplication
- Use GPU
- Solve heterogeneous systems and inequalities

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