SKEMA
BUSINESS SCHOOL

Introduction to
Artificial Intelligence

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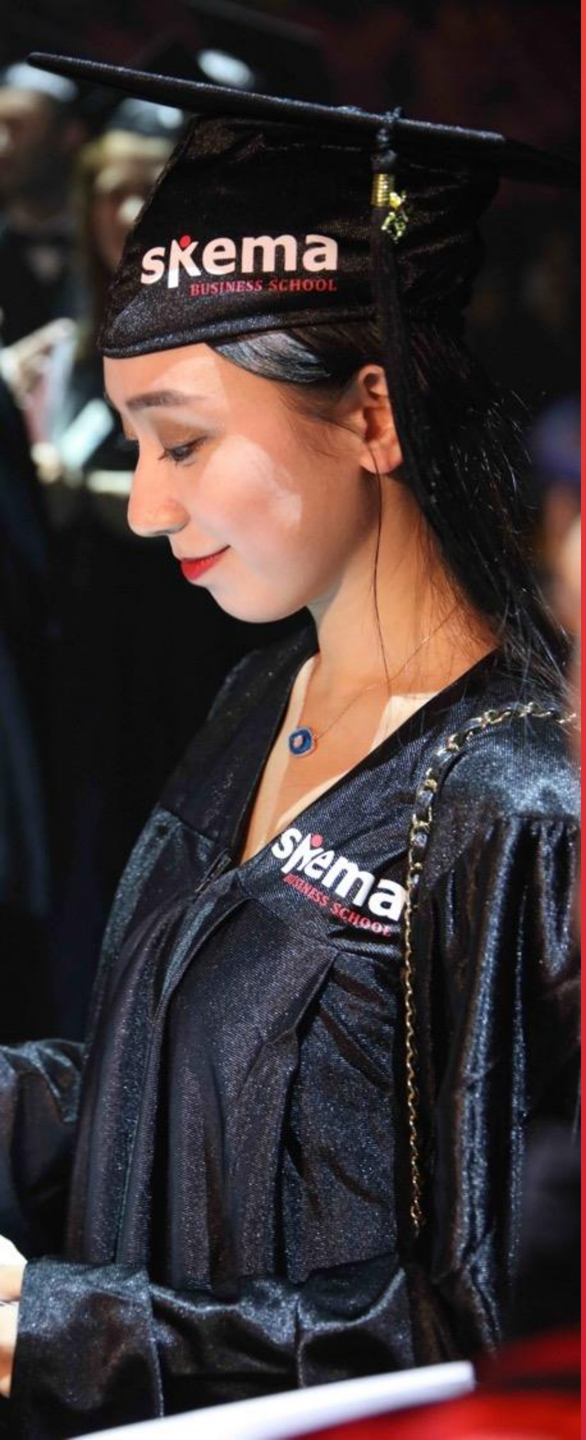
Lesson 6
Calculus of Predicates.
Automatic reasoning in Z3



Lesson 6. Calculus of Predicates. Automatic reasoning in Z3

Formal theory

Atoms, operations, formulae, inference rules, and axioms

Calculus of propositions

Negation and implication, tree axioms, and modus ponens

Logic of predicates

Domain, functions, predicates, and quantifiers.
Formalization of reasoning in natural language

Check reasoning with Z3

Z3 notation for predicate logic. Check of tautology and check of contradiction.

Formal theory

Atoms, operations, inference rules, axioms

$$\mathcal{L} = (\mathrm{A}, \Omega, \mathrm{Z}, \mathrm{I})$$

A – infinite countable set of variable or atomic formula

Ω – finite set of operator symbols

Z – finite set of inference rules

I – countable set of axioms

Theory defines a set of formula obtained from axioms applying inference rules

Propositional calculus as a formal theory

Atoms, operations, inference rules, axioms

$$\mathcal{L}_1 = (\mathrm{A}, \Omega, \mathrm{Z}, \mathrm{I}),$$

$$\mathrm{A} = \{p, q, r, s, t, u, p_1, \ldots\},$$

$$\Omega = \{\neg, \rightarrow\},$$

Z – modus ponens: $p \rightarrow \mathrm{q}, \mathrm{p} \vdash q$

$$\mathrm{I} = \{\mathrm{p} \rightarrow (\mathrm{q} \rightarrow \mathrm{p}),\ \big(\mathrm{p} \rightarrow (\mathrm{q} \rightarrow \mathrm{r})\big) \rightarrow \big((\mathrm{p} \rightarrow \mathrm{q}) \rightarrow (\mathrm{p} \rightarrow \mathrm{r})\big),\ (\neg\mathrm{p} \rightarrow \neg\mathrm{q}) \rightarrow (\mathrm{q} \rightarrow \mathrm{p})\}$$

Example of proof

Prove $A \to A$ in $\mathcal{L}_1$

1. $\mathrm{A} \to ((\mathrm{B} \to \mathrm{A}) \to \mathrm{A})$, axiom 1
2. $\left(\mathrm{A} \to ((\mathrm{B} \to \mathrm{A}) \to \mathrm{A})\right) \to \left((\mathrm{A} \to (\mathrm{B} \to \mathrm{A})) \to (\mathrm{A} \to \mathrm{A})\right)$, axiom 2
3. $\left((\mathrm{A} \to (\mathrm{B} \to \mathrm{A})) \to (\mathrm{A} \to \mathrm{A})\right)$, by modus ponens from 1 and 2
4. $\mathrm{A} \to (\mathrm{B} \to \mathrm{A})$, axiom 1
5. $\mathrm{A} \to A$, by modus ponens from 4 and 3

Predicate logic

First-order logic or quantificational logic

- Propositions, Predicates, and Quantifiers
- First-order logic uses quantified variables over non-logical objects
- Domain of discourse – an application area
- Predicate – a function that maps an object or cortege of objects, belonging to domain, into a set of logical constants True or False
- Quantifier symbols: $\forall$ for universal quantification, and $\exists$ for existential quantification

Predicates and quantifiers

Examples

- $P(x)$ – "x is smart"
- $Q(x,y)$ – "x loves y"
- $R(x,y,z)$ – "$x+y=z$"
- $\exists x\, P(x)$ – "someone is smart"
- $\forall y\, \exists x\, Q(x,y)$ – "whoever you are, someone loves you"

Z3 commands for predicate logic

Functions, predicates, and quantifiers

domain

(declare-sort U)

function

(declare-fun f (U) U)

predicate

(declare-fun g (U) Bool)

(declare-fun l (U U) Bool)

universal quantification

forall ((x U))

existential quantification

exists ((x U))

Formalize and check reasoning in Z3

- **All rabbits, that are not greedy, are black**

  No old rabbits are free from greediness

  Therefore: Some black rabbits are not old
- Domain – rabbits
- Predicates:

  G(x): "x is greedy"

  B(x): "x is black"

  O(x): "x is old"
- Formalization:

  $$\forall x\,\big(\neg G(x) \rightarrow B(x)\big), \forall x\,(O(x) \rightarrow G(x)) \vdash \exists x\,(B(x) \wedge \neg O(x))$$
- Prove contradiction of:

  $$\forall x\,\big(\neg G(x) \rightarrow B(x)\big), \forall x\,(O(x) \rightarrow G(x)), \neg(\exists x\,(B(x) \wedge \neg O(x)))$$

Z3 prove tautology

Conjunction of premises implication conclusion

```
(declare-sort U)
(declare-fun g (U) Bool)
(declare-fun b (U) Bool)
(declare-fun o (U) Bool)
(define-fun conjecture () Bool
   (=> (and
          (forall ((x U)) (=> (not (g x)) (b x) ))
          (forall ((x U)) (=> (o x) (g x) )))
       (exists ((x U)) (and (b x) (not (o x))))
   )
)
(assert (not conjecture))
(check-sat)
(get-model)
```

Z3 prove contradiction

Conjunction of premises and negation of conclusion

```
(declare-sort U)
(declare-fun g (U) Bool)
(declare-fun b (U) Bool)
(declare-fun o (U) Bool)
(define-fun conjecture () Bool
   (and  (forall ((x U)) (=> (not (g x)) (b x) ))
             (and (forall ((x U)) (=> (o x) (g x) ))
                       (not (exists ((x U)) (and (b x) (not (o x))))))
   )
)
(assert conjecture)
(check-sat)
(get-model
```

Z3 prove contradiction

Premises and negation of conclusion as assertions

```
(declare-sort U)
(declare-fun g (U) Bool)
(declare-fun b (U) Bool)
(declare-fun o (U) Bool)
(assert (forall ((x U)) (=> (not (g x)) (b x) )))
(assert (forall ((x U)) (=> (o x) (g x) )))
(assert (not (exists ((x U)) (and (b x) (not (o x))))))
(check-sat)
(get-model)
```

```
sat
(
  ;; universe for U:
  ;;   U!val!0
  ;; -----------
  ;; definitions for universe elements:
  (declare-fun U!val!0 () U)
  ;; cardinality constraint:
  (forall ((x U)) (= x U!val!0))
  ;; -----------
  (define-fun g ((x!0 U)) Bool
    true)
  (define-fun b ((x!0 U)) Bool
    false)
  (define-fun o ((x!0 U)) Bool
    false)
)
```

No contradiction observed

Study the model offered by Z3

The domain contains just one rabbit which is greedy, not black, and not old .

Let us extend this world

Find omitted premise (hypothesis)

Some reasoning looks credible though Z3 does not prove it

- some premises can be implied in a natural language narrative
- sometimes Z3 offers a trivial model to show satisfability
- analyzing the model try to extend specification adding premises
- for implications, sometimes it is implied that at least one representative exists that makes true the left part
- “All rabbits, that are not greedy, are black”
- implies that
- **“there is a not greedy rabbit”**
- add an assertion to observe unsatisfability

(assert (exists ((x U)) (not (g x))))

Check reasoning in logic of predicates

SYMBOLIC LOGIC By Lewis Carroll

- No shark ever doubts that it is well fitted out;
- A fish, that cannot dance a minuet, is contemptible;
- No fish is quite certain that it is well fitted out, unless it has three rows of teeth;
- All fishes, except sharks, are kind to children;
- No heavy fish can dance a minuet;
- A fish with three rows of teeth is not to be despised.
- Therefore: No heavy fish is unkind to children.

Formalize statements in natural language

Introduce predicate and use them with quantifiers and logical connectives

- “all fish, except sharks, are kind to children”
- $F(x)$ – “x is fish”
- $S(x)$ – “x is shark”
- $C(x)$ – “x is a child”
- $K(x,y)$ – “x is kind to y”
- $\forall x\ \forall y\ F(x) \land (\neg S(x)) \land C(y) \rightarrow K(x,y)$

Task 2 – check reasoning in predicate logic

Use Z3

- Choose and specify domain
- Denote predicates
- Write premises and conclusion as formulae
- Represent reasoning as Z3 program
- Check reasoning in Z3

Variants of task 2

1. All soldiers are strong. All soldiers are brave. ⊢ Some strong men are brave.
2. I admire these pictures. When I admire anything I wish to examine it thoroughly. ⊢ I wish to examine some of these pictures thoroughly.
3. None but the brave deserve the fair. Some braggarts are cowards. ⊢ Some braggarts do not deserve the fair.
4. All soldiers can march. Some babies are not soldiers. ⊢ Some babies cannot march.
5. All selfish men are unpopular. All obliging men are popular. ⊢ All obliging men are unselfish.
6. Some epicures are ungenerous. All my uncles are generous. ⊢ My uncles are not epicures.
7. No misers are unselfish. None but misers save egg-shells. ⊢ No unselfish people save egg-shells.
8. All, who are anxious to learn, work hard. Some of these boys work hard. ⊢ Some of these boys are anxious to learn.
9. All lions are fierce. Some lions do not drink coffee. ⊢ Some creatures that drink coffee are not fierce.
10. No Professors are ignorant. All ignorant people are vain. ⊢ No professors are vain.

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