VERIFICATION OF ETHERNET PROTOCOLS VIA PARAMETRIC COMPOSITION OF PETRI NET

Zaitsev D.A., Zaitsev I.D.

Dmitry A. Zaitsev
National Telecommunication Academy
Kuznechnaya, 1, Odessa 65029 Ukraine
http://www.geocities.com/zsoftua

Ivan D. Zaitsev
Novosibirsk State University
Pirogova, 2, Novosibirsk 630090 Russia
ivan_d_zaitsev@mail.ru

Abstract: The verification of Ethernet protocols represented by Petri net models is implemented. A model consisting of an arbitrary number of workstations on bus is composed out of model of a single workstation via the fusion of contact places. The invariance of the model with an arbitrary number of workstations is proved on the base of the invariance of separate workstation's model, which is the functional subnet. Parametric equations and invariants are used.

Keywords: Ethernet, Petri net, invariant, composition, functional subnet

1. INTRODUCTION

The verification of protocols (Marsan, *et al.*, 1987) is the traditional area of Petri nets (Girault and Valk, 2003) application. Really, the majority of network protocols assume an asynchronous character of systems' interaction, which makes its description with sequential models such as, for instance, flow block, difficult.

For the investigation of protocols, as a rule, two basic tasks are solved: correctness proof and performance evaluation. The first of the above mentioned tasks is named also as verification of protocol. It is the highest priority indeed, since the presence of defects in source specifications constitutes the most expensive type of errors. Since in the case an incorrect protocol would be implemented into either software or hardware, the expenses concerned with the debugging became enormously huge. It was shown (Girault and Valk, 2003) that Petri net model of a correct telecommunication protocol has to be invariant one.

The goal of the present work is to prove the invariance of the local area network Ethernet protocols with the bus topology for an arbitrary number of workstations. In the presence of early known detailed models of Ethernet (Marsan, *et al.*, 1987), formal proof has not been implemented due to the enormous dimension of Petri net. The application of decomposition technique (Zaitsev, 2004a) with properties of functional subnets' invariants (Zaitsev, 2004b, 2005) usage allowed the solution of the mentioned task. Notice that, the class of functional subnets essentially distinguishes from the subnets studied in (Christensen and Petrucci, 2000; Haddad, *at al.*, 2002; Juan, *at al.*, 1998).

Moreover, as distinct from (Zaitsev, 2004c, 2004d), the parametric approach is applied. On the base of the regular structure of the model, composed as the repetition of the model of workstation, we prove the invariance for an arbitrary number of workstations. Invariants are obtained in symbolic form of algebraic expression, which allows its easy calculation for concrete values of parameters. This approach may be applied successfully for large-scale systems with regular structure, such as, for instance, computer hardware and software (Cortadella, *at al.*, 2003).

2. PROTOCOLS OF ETHERNET

In Ethernet network the protocols of multiple media access with carrier sense and collision detection (CSMA/CD) are implemented. Source specifications of protocols are regulated by standards IEEE 802.3.

Each of workstations listens to the media and in the case of carrier absence may start the transmission of data. If several workstations start the transmission of data simultaneously, then information overlapping occurs, that is called a collision. The workstation has facilities to detect collisions. In the case the collision being detected, the transmission of data is broken and then is resumed in a random interval of time.

The specifications of protocols regulate also the format of data (frames) transmitting. In the present work the structure of Ethernet data frame is not used and, moreover, the network with bus topology is investigated. The start and finish of frame's transmission and the processes of carrier spreading on media in time are considered as well as the collisions' detection and recognition. Notice that, carrier appearance and cessation spread on the bus during the time interval, so the workstations have got different information about channel's state in the same chosen moment of time. In spite of Ethernet with bus topology is replaced by switched Ethernet recently, CSMA/CD procedures are actual concerning radio Ethernet and other modern wireless technologies.

3. MODEL OF ETHERNET

In the present work the model of Ethernet with bus topology and an arbitrary number of workstations attached is investigated. It constitutes slightly modified model described in Marsan's work (Marsan, *et al.*, 1987). In the mentioned work, it was pointed to the constructed model is detailed and precise enough implementation of source specifications. But due to the huge dimension even for the minor number of workstations the investigation of the model is too hard. That is why, it was suggested in (Marsan, *et al.*, 1987) to investigate the simplified model.

The Petri net composition technique application allows the verification of detailed model, including an arbitrary number of workstations. Marshan's source model is modified (Fig. 1) in such a way that the model of separate workstation constitutes the functional Petri net (Zaitsev, 2004a, 2005). For these purposes contact places were added. The composition of local area network model was implemented by the workstations' contact places fusion. Thus, there is the natural decomposition of the entire model into functional subnets.

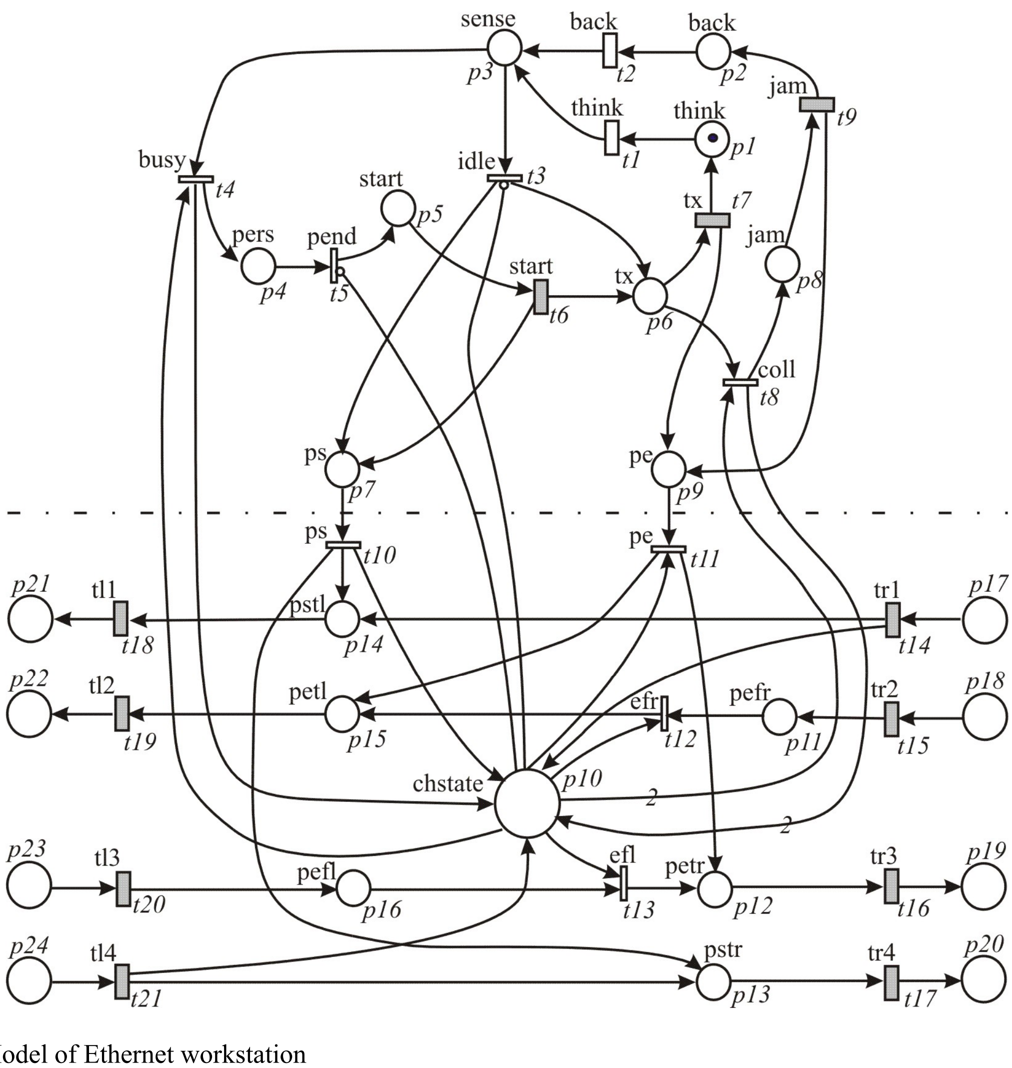

Fig. 1. Model of Ethernet workstation

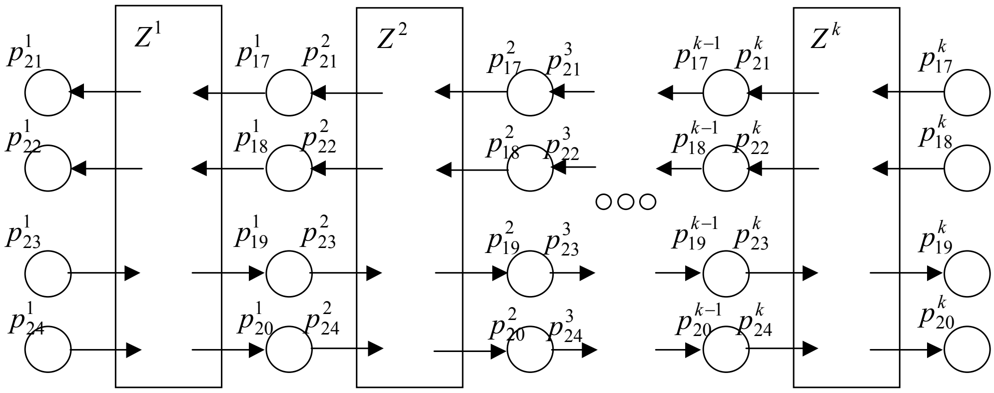

Fig. 2. Model of Ethernet with bus topology

The model of workstation (Fig. 1) has input places $X=\{p_{17},p_{18},p_{23},p_{24}\}$ and output places $Y=\{p_{19},p_{20},p_{21},p_{22}\}$ and according to the definition (Zaitsev, 2004a, 2005) is a functional Petri net. The model is represented in the form of timed Petri net with inhibitor arcs (Girault and Valk, 2003). Notice that, transitions of three types were used: firing instantly represented by narrow bars; firing in deterministic time, represented by filled bars; and firing in random uniformly distributed time, represented by wide unfilled bars. The inhibitor arcs are used for zero marking check. Instead of the arrow such arcs contain the little circle. The contact places are drawn in larger size.

4. COMPOSING MODELS OF NETWORK

Let us consider the rules of model construction for Ethernet network with bus topology consisting of *k* workstations. The general scheme of such a model is represented in Fig. 2. The models of workstations Z^i are combined by means of contact places' fusion. At that, each workstation interacts exactly with two neighboring workstations. Let's consider, for instance, subnet Z^2. Contact places $p_{21},p_{22},p_{23},p_{24}$ of subnet Z^2 are merged with contact places $p_{17},p_{18},p_{19},p_{20}$ of subnet Z^1 correspondingly. Moreover, contact places $p_{17},p_{18},p_{19},p_{20}$ of subnet Z^2 are merged with contact places $p_{21},p_{22},p_{23},p_{24}$ of subnet Z^3 correspondingly. The belonging of elements to concrete subnet is represented with the aid of an upper index equaling to the subnet's number.

Notice that, the constructed model has natural decomposition into functional subnets $\{Z^i\}$. This fact will be used further for model's properties analysis.

5. INVARIANCE OF MODEL

As was mentioned in (Marsan, *at al.*, 1987; Girault and Valk, 2003), an ideal model of a correct telecommunication protocol has to possess such properties as boundness, safeness, liveness.

To determine the mentioned properties Petri net invariants are applied (Girault and Valk, 2003). In this case the model of a correct protocol ought to be invariant one. Since known methods of invariants calculation (Kryviy, 1999) have exponential complexity, their application becomes practically impossible in the investigation of large-scale nets.

In (Zaitsev, 2004b) the technique of Petri net invariants calculation on the base of invariants of its functional subnets was presented. For investigation of an arbitrary given Petri net its decomposition into functional subnets is implemented preliminarily. The technique of decomposition was described in (Zaitsev, 2004a, 2005).

Table 1. Basis invariants of workstation's model

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	1	1		1		1			1		1		1			1		1
2												1				1			1				1	
3											1				1			1				1		
4						1		1	1															
5														1			1				1			
6													1							1				1

As set $\{Z^i\}$ for the constructed model of Ethernet with bus topology defines the partition of Petri net into functional subnets, so the method described in (Zaitsev, 2004b, 2005) may be applied for invariants calculation. Remember that, the invariant is a nonnegative integer solution $\bar{x}$ of equation

$$\bar{x} \cdot A = 0 , \qquad (1)$$

where A is the incidence matrix of Petri net for invariants of places or the transposed incidence matrix for invariants of transitions.

Notice that, in the constructed Ethernet model all the functional subnets have the same structure and are distinct only in the upper index of their elements. At calculation of subnet Z^i invariants the inhibitor arcs and loops were not been taken into consideration, as they do not change the marking of net. Moreover, arcs (t_3, p_6), (t_6, p_6), (t_{10}, p_{13}), (t_{10}, p_{14}), (t_{11}, p_{12}), (t_{11}, p_{15}) were omitted and arc (t_{11}, p_6) was added, as the source model is not invariant. These transformations of matrix correspond to the starting of frame's retransmission only after cessation signal have been put into the bus. Thus, the following incidence matrix is used:

$$A = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

The application of tool Tina (Berthomieu, *at al.*, 2004) gives us the matrix of basis solutions for invariants of places represented in Table 1. Row l of the represented matrix is the basis invariant, which at construction of general solution is multiplied by any nonnegative integer coefficient z_l. The columns of matrix correspond to places of net Z^i.

According to (Zaitsev, 2004b), after calculation of functional subnets' invariants it is necessary to find the common invariants for contact places. As in the composition of two neighbor subnets Z^i and Z^{i+1} four contact places $\{p_{17}^i, p_{18}^i, p_{19}^i, p_{20}^i\}$ of the left subnet and four contact places $\{p_{21}^{i+1}, p_{22}^{i+1}, p_{23}^{i+1}, p_{24}^{i+1}\}$ of the right subnet take part, so the system of equation for contact places has the following form:

$$\begin{cases} z_1^i + z_5^i = z_5^{i+1}, \\ z_3^i = z_1^{i+1} + z_3^{i+1}, \\ z_1^i + z_2^i = z_2^{i+1}, \\ z_6^i = z_1^{i+1} + z_6^{i+1}, \quad i = \overline{1, k-1}. \end{cases} \qquad (2)$$

It should be noted, that system (2) contains an infinite number of equations. For its solution the sequence of concrete systems for values $k = 2,3,...$ was constructed. Then, these systems are solved using Tina (Berthomieu, *at al.*, 2004) tool. The application of inductive reasoning gives us the parametric form of solutions' representation (see Appendix). System (2) has the following $2 \cdot k + 4$ basis parametric solutions:

$$\begin{pmatrix} (z_4^i), \quad i = \overline{1,k}; \\ (z_2^j, \quad j = \overline{1,k}); \\ (z_3^j, \quad j = \overline{1,k}); \\ (z_5^j, \quad j = \overline{1,k}); \\ (z_6^j, \quad j = \overline{1,k}); \\ \begin{pmatrix} ((z_3^j, z_6^j), \quad j = \overline{1,i-1}), \quad (z_1^i), \\ ((z_2^j, z_5^j), \quad j = \overline{i+1,k}) \end{pmatrix}, \quad i = \overline{1,k} \end{pmatrix}. \qquad (3)$$

Notice that, the first and sixth solutions of (3) represent k concrete solutions of system; moreover, solutions are described by indication of nonzero (equalling to unit) elements.

Let us substitute the basis invariants, corresponding to free variables z_l of Table 1, into parametric solutions (3) of system for contact places (2). We obtain the following parametric solutions of the source system (1):

$$\begin{pmatrix} (p_6^i, p_8^i, p_9^i), \quad i = \overline{1,k}; \\ ((p_{12}^j, p_{16}^j, p_{19}^j, p_{23}^{j=1}), \quad j = \overline{1,k}); \\ ((p_{11}^j, p_{15}^j, p_{18}^j, p_{22}^{j=1}), \quad j = \overline{1,k}); \\ ((p_{14}^j, p_{17}^j, p_{21}^{j=1}), \quad j = \overline{1,k}); \\ ((p_{13}^j, p_{20}^j, p_{24}^{j=1}), \quad j = \overline{1,k}); \\ \begin{pmatrix} ((p_{11}^j, p_{15}^j, p_{18}^j, p_{20}^j, p_{22}^{j=1}, p_{24}^{j=1}), \\ j = \overline{1,i-1}), \\ (p_1^i, p_2^i, p_3^i, p_4^i, p_5^i, p_6^i, p_7^i, p_8^i, \\ p_{10}^i, p_{12}^i, p_{15}^i, p_{17}^i, p_{19}^i, p_{22}^{i=1}, p_{24}^{i=1}), \\ ((p_{12}^j, p_{14}^j, p_{16}^j, p_{17}^j, p_{19}^j, \\ p_{21}^{j=1}, p_{22}^{j=1}, p_{24}^{j=1}), j = \overline{i+1,k}) \end{pmatrix}, i = \overline{1,k} \end{pmatrix} \qquad (4)$$

In the description of solutions conditional elements are indicated, which are present only in the first subnet, because only it has contact places $p_{21}, p_{22}, p_{23}, p_{24}$; in the fusion of places for the others workstations their minor numbers in the left neighbor subnet $p_{17}, p_{18}, p_{19}, p_{20}$ correspondingly are used.

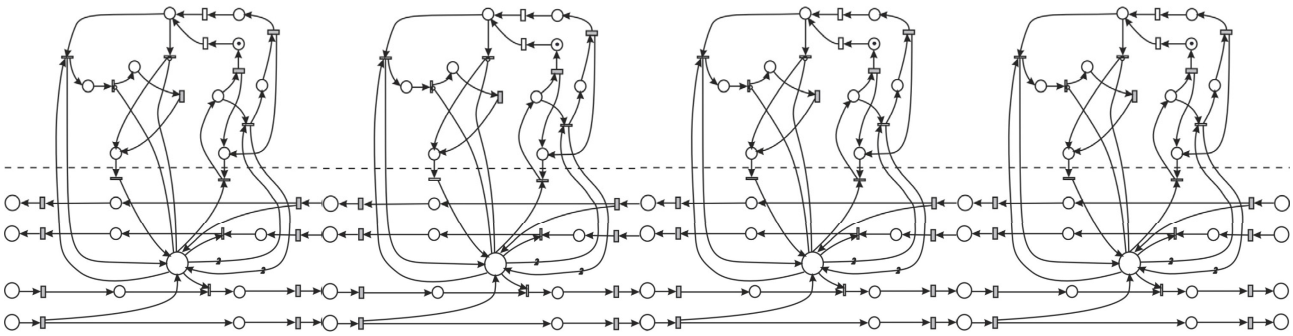

Fig. 3. Model of bus with four workstations

Since, for instance, at choice the sum of all the basis solutions, the natural invariant of the entire net is obtained, the model of Ethernet with the bus topology and an arbitrary number k of workstations is a p-invariant Petri net.

To verify the obtained parametric solution the models for various numbers of workstations on the bus were generated and their invariants were calculated with the aid of tool Tina (Berthomieu, *at al.*, 2004). The results coincide. For example, for the model with four workstations (Fig. 3) the following invariants were obtained:

$(p_6^1, p_8^1, p_9^1);$

$(p_6^2, p_8^2, p_9^2);$

$(p_6^3, p_8^3, p_9^3);\,;$

$(p_6^4, p_8^4, p_9^4);$

$(p_{12}^1, p_{16}^1, p_{19}^1, p_{23}^1, p_{12}^2, p_{16}^2, p_{19}^2, p_{12}^3, p_{16}^3, p_{19}^3, p_{12}^4, p_{16}^4, p_{19}^4);$

$(p_{11}^1, p_{15}^1, p_{18}^1, p_{22}^1, p_{11}^2, p_{15}^2, p_{18}^2, p_{11}^3, p_{15}^3, p_{18}^3, p_{11}^4, p_{15}^4, p_{18}^4);$

$(p_{14}^1, p_{17}^1, p_{21}^1, p_{14}^2, p_{17}^2, p_{14}^3, p_{17}^3, p_{14}^4, p_{17}^4);$

$(p_{13}^1, p_{20}^1, p_{24}^1, p_{13}^2, p_{20}^2, p_{13}^3, p_{20}^3, p_{13}^4, p_{20}^4);$

$(p_1^1, p_3^1, p_3^1, p_4^1, p_5^1, p_6^1, p_7^1, p_8^1, p_{10}^1, p_{12}^1, p_{15}^1, p_{17}^1, p_{19}^1, p_{22}^1, p_{24}^1,$
$p_{12}^2, p_{14}^2, p_{16}^2, p_{17}^2, p_{19}^2, p_{12}^3, p_{14}^3, p_{16}^3, p_{17}^3, p_{19}^3, p_{12}^4, p_{14}^4, p_{16}^4, p_{17}^4, p_{19}^4);$

$(p_{11}^1, p_{13}^1, p_{15}^1, p_{18}^1, p_{20}^1, p_{22}^1, p_{24}^1, p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, p_6^2, p_7^2, p_8^2,$
$p_{10}^2, p_{12}^2, p_{15}^2, p_{17}^2, p_{19}^2, p_{12}^3, p_{14}^3, p_{16}^3, p_{17}^3, p_{19}^3, p_{12}^4, p_{14}^4, p_{16}^4, p_{17}^4, p_{19}^4);$

$(p_{11}^1, p_{13}^1, p_{15}^1, p_{18}^1, p_{20}^1, p_{22}^1, p_{24}^1, p_{11}^2, p_{13}^2, p_{15}^2, p_{18}^2, p_{20}^2, p_1^3, p_2^3, p_3^3,$
$p_4^3, p_5^3, p_6^3, p_7^3, p_8^3, p_{10}^3, p_{12}^3, p_{15}^3, p_{17}^3, p_{19}^3, p_{12}^4, p_{14}^4, p_{16}^4, p_{17}^4, p_{19}^4);$

$(p_{11}^1, p_{13}^1, p_{15}^1, p_{18}^1, p_{20}^1, p_{22}^1, p_{24}^1, p_{11}^2, p_{13}^2, p_{15}^2, p_{18}^2, p_{20}^2, p_{11}^3, p_{13}^3, p_{15}^3,$
$p_{18}^3, p_{20}^3, p_1^4, p_2^4, p_3^4, p_4^4, p_5^4, p_6^4, p_7^4, p_8^4, p_{10}^4, p_{12}^4, p_{15}^4, p_{17}^4, p_{19}^4).$

These invariants may be generated easily with the aid of the parametric basis (4).

7. CONCLUSION

Thus, in the present work the invariance of the Petri net model of local area network Ethernet with bus topology was proved. The model of Ethernet constitutes the composition of functional subnets modeling workstations. The invariance of the general model was proved on the base of invariance of functional subnets. The application of composition technique at invariants investigation allows the analysis of protocols for an arbitrary number of workstations in the network. To handle invariants for an arbitrary number of workstations the parametric form of representation was used.

The parametric composition described may be applied successfully for objects with a regular structure, for instance, in hardware and software design.

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APPENDIX
SOLUTIONS OF PARAMETRIC SYSTEM

I. Sequence of systems:

$$k=2:\begin{cases} z_1^1+z_5^1=z_5^2, \\ z_3^1=z_1^2+z_3^2, \\ z_1^1+z_2^1=z_2^2, \\ z_6^1=z_1^2+z_6^2. \end{cases}$$

$$k=3:\begin{cases} z_1^1+z_5^1=z_5^2, \\ z_3^1=z_1^2+z_3^2, \\ z_1^1+z_2^1=z_2^2, \\ z_6^1=z_1^2+z_6^2, \\ z_1^2+z_5^2=z_5^3, \\ z_3^2=z_1^3+z_3^3, \\ z_1^2+z_2^2=z_2^3, \\ z_6^2=z_1^3+z_6^3. \end{cases}$$

...

II. Sequence of solutions:

k	Basis solutions
2	(z_4^1), (z_4^2), (z_2^1,z_2^2), (z_3^1,z_3^2), (z_5^1,z_5^2), (z_6^1,z_6^2), (z_3^1,z_6^1,z_1^2), (z_1^1,z_2^2,z_5^2).
3	(z_4^1), (z_4^2), (z_4^3), (z_2^1,z_2^2,z_2^3), (z_3^1,z_3^2,z_3^3), (z_5^1,z_5^2,z_5^3), (z_6^1,z_6^2,z_6^3), $(z_3^1,z_6^1,z_1^2,z_2^3,z_5^3)$, $(z_1^1,z_2^2,z_5^2,z_2^3,z_5^3)$, $(z_3^1,z_6^1,z_3^2,z_6^2,z_1^3)$.
4	(z_4^1), (z_4^2), (z_4^3), (z_4^4), $(z_2^1,z_2^2,z_2^3,z_2^4)$, $(z_3^1,z_3^2,z_3^3,z_3^4)$, $(z_5^1,z_5^2,z_5^3,z_5^4)$, $(z_6^1,z_6^2,z_6^3,z_6^4)$, $(z_3^1,z_6^1,z_1^2,z_2^3,z_5^3,z_2^4,z_5^4)$, $(z_3^1,z_6^1,z_3^2,z_6^2,z_1^3,z_2^4,z_5^4)$, $(z_3^1,z_6^1,z_3^2,z_6^2,z_3^3,z_6^3,z_1^4)$, $(z_1^1,z_2^2,z_5^2,z_2^3,z_5^3,z_2^4,z_5^4)$.
5	(z_4^1), (z_4^2), (z_4^3), (z_4^4), (z_4^5), $(z_2^1,z_2^2,z_2^3,z_2^4,z_2^5)$, $(z_3^1,z_3^2,z_3^3,z_3^4,z_3^5)$, $(z_5^1,z_5^2,z_5^3,z_5^4,z_5^5)$, $(z_6^1,z_6^2,z_6^3,z_6^4,z_6^5)$, $(z_3^1,z_6^1,z_1^2,z_2^3,z_5^3,z_2^4,z_5^4,z_2^5,z_5^5)$, $(z_3^1,z_6^1,z_3^2,z_6^2,z_1^3,z_2^4,z_5^4,z_2^5,z_5^5)$, $(z_3^1,z_6^1,z_3^2,z_6^2,z_3^3,z_6^3,z_1^4,z_2^5,z_5^5)$, $(z_3^1,z_6^1,z_3^2,z_6^2,z_3^3,z_6^3,z_3^4,z_6^4,z_1^5)$, $(z_1^1,z_2^2,z_5^2,z_2^3,z_5^3,z_2^4,z_5^4,z_2^5,z_5^5)$.