COMPOSITIONAL ANALYSIS OF PETRI NETS

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Foundations of compositional analysis of Petri nets are presented. This analysis consist of the determination of properties of a given Petri net from the properties of its functional subnets. Compositional analysis covers the investigation of behavioral and structural properties of Petri nets with the help of matrix methods that use fundamental equations and invariants. The exponential acceleration of computations as a function of the dimensionality of a net is obtained.

Keywords: *Petri net, functional subnet, composition.*

INTRODUCTION

Petri nets [1, 2] are successfully used in investigating systems and processes in various applied domains [3, 4]. As a rule, a model of a real object has a sufficiently high dimensionality and consists of more than a thousand elements. At the same time, basic methods of analysis of properties of Petri nets except for, perhaps, reduction methods have the exponential computational complexity. Thus, the development of efficient methods for the investigation of properties of Petri nets is a vital scientific issue.

The following three groups of methods of analyzing properties of Petri nets are well known [2]: the methods based on the construction of trees of attainable and covering markings, matrix methods using the fundamental equation of a net and invariants, and reduction methods. It is relevant to note that reduction is an auxiliary means of investigation and a special case of equivalent transformations [5] decreasing the dimensionality of a net.

Matrix methods are most promising for the analysis of large technical systems [3]. The fundamental equation of a Petri net is a system of linear Diophantine equations [2]. The solutions of such a system are interpreted as vectors of calculation of firings of allowable sequences of transitions and, hence, must be nonnegative integer numbers, which stipulates the specifity of the problem. Methods of solution of such systems are presented in [6–8]. Unfortunately, all the well-known methods have asymptotically exponential complexity, which complicates their use in analyzing real systems.

The objective of this article is the construction of compositional methods that make it possible to analyze properties of Petri nets and to substantially accelerate calculations. In fact, models of complex systems are constructed, as a rule, from models of their components. Thus, it is necessary to formalize this process and to construct methods that allow one to find properties of the entire net on the basis of known properties of its subnets.

Moreover, in the cases when the composition of a model in terms of its subnets is not given, one can use the methods of decomposition of Petri nets that are presented in [9, 10]. A decomposition algorithm makes it possible to partition a given Petri net into a set of minimal functional subnets. In this article, it is shown how the properties of functional subnets that specify a partition of the initial Petri net can be used for computation of properties of the entire net. The obtained acceleration of computations is estimated by an exponential function. Since the dimensionality of subnets is, as a rule, substantially less than the dimensionality of the entire net, the actual acceleration of computations can be very sizable, which is substantiated by the results of application of this method to the analysis of net protocols [11, 12].

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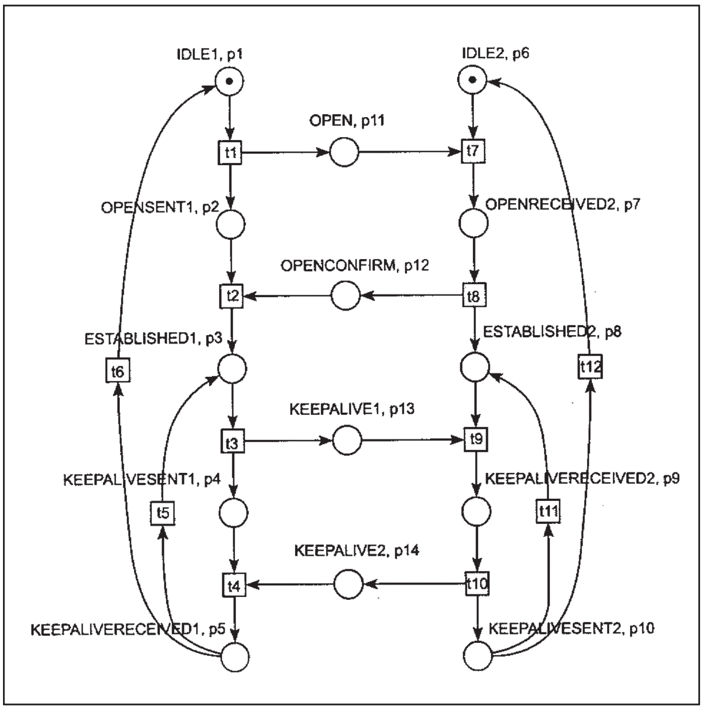

Fig. 1. A Petri model of the BGP protocol.

MATRIX METHODS OF INVESTIGATION OF PROPERTIES OF PETRI NETS

A Petri net is a quadruple $N=(P,T,F,W)$, where $P=\{p\}$ is a finite set of nodes called places, $T=\{t\}$ is a finite set of nodes called transitions, the adjacency relation of nodes $F \subseteq P\times T \cup T\times P$ specifies the set of edges that connect places and transitions, a mapping $W:F\to \mathbb{N}$ specifies the multiplicity of edges, and $\mathbb{N}$ denotes the set of natural numbers. In Fig. 1, a Petri net that models the BGP protocol is presented [13].

We denote the sets of input and output nodes for places and transitions of such a net as follows:

$$\bullet p=\{t|\exists(t,p)\in F\},\ p\bullet=\{t|\exists(p,t)\in F\};\ \bullet t=\{p|\exists(p,t)\in F\},\ t\bullet=\{p|\exists(t,p)\in F\}.$$

A marking of a net is understood to be a mapping $\mu:P\to\mathbb{N}_0$ specifying a distribution of dynamic elements called tokens among places, where $\mathbb{N}_0$ is the set of nonnegative integer numbers. A marked Petri net is a pair $M=(N,\mu_0)$ or a quintuple $M=(P,T,F,W,\mu_0)$, where μ_0 is the initial marking.

The fundamental equation of a Petri net [2] is of the form

$$\bar{\mu}=\bar{\mu}_0+A\bar{y} \text{ or } \bar{y}A^T=\Delta\bar{\mu}, \tag{1}$$

where A is the incidence matrix of the Petri net, $\bar{y}$ is the vector of calculation of firings of transitions, and $\Delta\bar{\mu}=\bar{\mu}-\bar{\mu}_0$. The solvability of a fundamental equation in nonnegative integers is the necessary condition of the attainability of a given marking [2, 3].

By a p-invariant of a Petri net [2] we understand nonnegative integer solutions of the system

$$\bar{x}A=0. \tag{2}$$

By a t-invariant we understand nonnegative integer solutions of system (2) with the transposed incidence matrix. A net is invariant if it has an invariant in which all components are natural numbers. Invariants play the key role in investigating properties of Petri nets such as boundedness, safety, and liveness [2, 3, 14].

The majority of well-known problems of analyzing properties of Petri nets [2, 3, 4] are reduced to the solution of systems of linear equations and inequalities. The necessary and sufficient conditions for the basic structural properties are presented in Table 1. Moreover, {0, 1} solutions of systems of inequalities with modified incidence matrices are characteristic vectors of siphons and traps used in analyzing properties of nets with free choice [1, 2, 15].

TABLE 1. Necessary and Sufficient Conditions for Structural Properties of Petri Pets

Structural Property of Petri Nets	Necessary and Sufficient Conditions
Boundedness	$\exists \bar{x} > 0,\ \bar{x}A \le 0$
Safety	$\exists \bar{x} > 0,\ \bar{x}A = 0$
Repetitiveness	$\exists\, \bar{y} > 0,\ A\bar{y} \ge 0$
Constancy	$\exists\, \bar{y} > 0,\ A\bar{y} = 0$

As is shown in [7, 16], a homogeneous system containing equations and inequalities can be reduced to a homogeneous system of equations. We note that to the mentioned transformations corresponds a modification of the initial net in such a manner that the finding of some property or other can be considered as the determination of p-invariants of the modified net. Therefore, in what follows, without loss of generality, we will solve homogeneous equation (2) to find structural properties and nonhomogeneous equation (1) to find behavioral properties of Petri nets.

Moreover, according to [7], we represent the general solution of a homogeneous system as a linear combination with nonnegative integer coefficients of basic solutions. The basis consists of solutions minimal in the nonnegative integer lattice of the system. In contrast to the classical theory of linear systems, in order to represent the general solution of a nonhomogeneous system in nonnegative integer numbers, it is necessary to use not an arbitrary solution but a set of minimal particular solutions.

FUNCTIONAL SUBNETS

By a net with input and output places we understand a Petri net in which special subsets of places are specified, namely, input and output ones.

We call a functional net a triple $Z = (N, X, Y)$, where N is a Petri net, $X \subseteq P$ are its input places, $Y \subseteq P$ are its output places, and the sets of input and output places do not intersect, $X \cap Y = \varnothing$; moreover, the input places have no incoming edges and output places have no outgoing ones, i.e., we have $\forall p \in X : \bullet p = \varnothing$ and $\forall p \in Y : p \bullet = \varnothing$. We call the places from the set $Q = P \setminus (X \cup Y)$ internal and the places $C = X \cup Y$ contact.

A Petri net $N' = (P', T', F')$ is a subnet of a net N if we have $P' \subseteq P$, $T' \subseteq T$, and $F' \subseteq F$.

We call a functional net $Z = (N', X, Y)$ a functional subnet of a net N and denote it by $Z \succ N$ if N' is a subnet of N and, moreover, Z is connected with the rest of the net only by edges incident to input or output places; in this case, the input places can have only incoming edges and the output places can have only outgoing edges. Thus, we have

$$\forall p \in X : \{(p, t) | t \in T \setminus T'\} = \varnothing, \quad \forall p \in Y : \{(t, p) | t \in T \setminus T'\} = \varnothing,$$

$$\forall p \in Q : \{(p, t) | t \in T \setminus T'\} = \varnothing \wedge \{(t, p) | t \in T \setminus T'\} = \varnothing.$$

A functional subnet $Z' \succ N$ is minimal if and only if it does not contain another functional subnet of the initial Petri net N.

We denote a net generated by a given set of transitions $R \subseteq T$ by $B(R)$. We call a subnet $Z = B(R) = (X, Q, Y, R)$ of a Petri net N complete in N if the relations $X^{\bullet} \subseteq R$, ${}^{\bullet}Y \subseteq R$, and ${}^{\bullet}Q^{\bullet} \subseteq R$ are true in N.

Properties of functional subnets are investigated in [10, 17]. Among them, the following properties are most important:

(1) a functional subnet is generated by the set of its transitions;

(2) the set of minimal functional subnets $\mathfrak{F} = \{Z^j\}$, $Z^j \succ N$, determines a partition of the set T into disjoint subsets T^j such that $T = \bigcup_j T^j$ and $T^j \cap T^k = \varnothing$, $j \ne k$;

(3) any functional subnet Z' of an arbitrary Petri net N is the sum (union) of a finite number of minimal functional subnets; the union of subnets can also be defined with the help of the operation of merging contact places.

In [10], two methods of decomposition of a given Petri net into functional subnets are presented and substantiated, namely, with the help of logical equations and with the help of a specially proposed algorithm. The complexity of the algorithm is demonstrated to be polynomial and equal to the cube of the number of nodes of a net. A decomposition of the mentioned model of the BGP protocol (Fig. 1) into functional subnets is presented in Fig. 2. Note that they are not minimal.

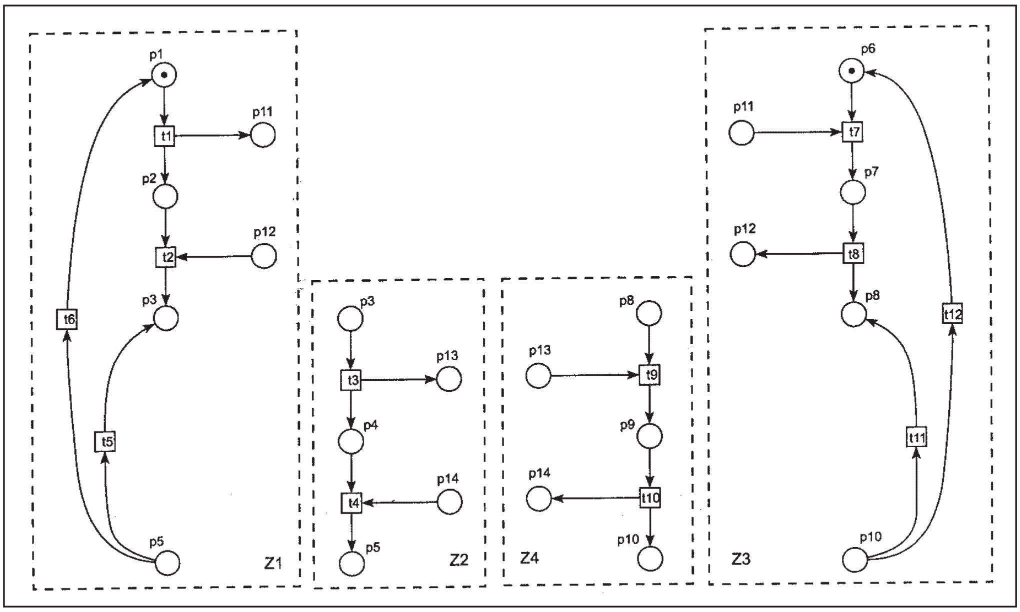

Fig. 2. Decomposition of the model of the BGP protocol.

To describe interrelations between functional subnets, a graph of functional subnets is introduced in [10]. In this article, a net of functional subnets is introduced. Such a net is demonstrated to be a marked graph, i.e., its places have no more than one incoming edge and no more than one outgoing edge. This property underlies the compositional analysis of Petri nets. In a net of functional subnets, the set of places consists of the contact places of the initial net and transitions are minimal functional subnets. Let us formally define such a net.

By the net of functional subnets of a given Petri net N we understand a Petri net N' such that we have $P' = C$, $T' = \{t^Z \mid t^Z \leftrightarrow Z,\ Z \succ N\}$, $(p', t^Z) \in F' \Leftrightarrow \exists t \in Z: (p', t) \in F$, $(t^Z, p') \in F' \Leftrightarrow \exists\, t \in Z: (t, p') \in F$, and $W' = W$. We note that a net of functional subnets can also be defined for some decomposition containing functional subnets that are not minimal.

THEOREM 1. In a decomposition of a Petri net into minimal functional subnets, a contact place is incident to no more than one input and to no more than one minimal functional output subnet.

Proof. Suppose the contrary. Let us consider each possible variant separately, namely,

(1) there exists a contact place $p \in C$ that has several minimal input functional subnets;

(2) there exists a contact place $p \in C$ that has several minimal functional output subnets.

In case (1), there are minimal functional nets Z' and Z'' such that we have

$$(\exists t' \in Z',\ t' \in {}^{\bullet}p) \wedge (\exists t'' \in Z'',\ t'' \in {}^{\bullet}p).$$

Since, according to Theorem 4 from [10], each minimal functional subnet is complete in N, transitions t' and t'' belong to one minimal functional subnet according to the definition of completeness. Thus, we obtain a contradiction.

In case (2), there are minimal functional nets Z', Z'' such that we have

$$(\exists t' \in Z',\ t' \in p\bullet) \wedge (\exists t'' \in Z'',\ t'' \in p\bullet).$$

As in case (1), we have a contradiction.

The contradiction obtained proves the theorem.

COROLLARY 1. A net of functional subnets is a marked graph.

In Fig. 3, the net of functional subnets is presented that corresponds to the decomposition depicted in Fig. 2 for the model of the BGP protocol (Fig. 1).

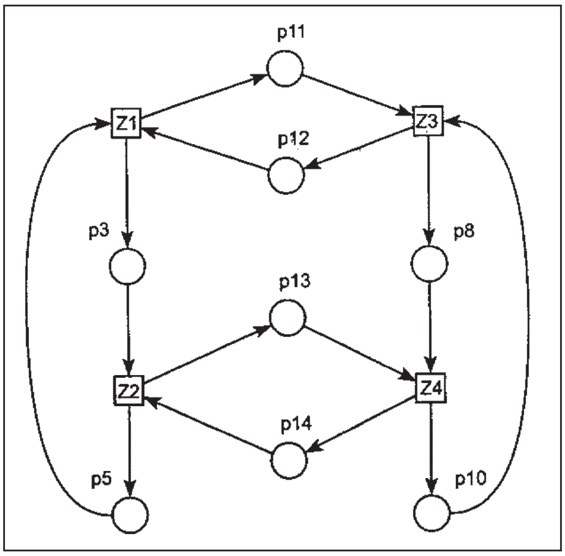

Fig. 3. The net of functional subnets of the model of BGP.

PROPERTIES OF INVARIANTS OF FUNCTIONAL SUBNETS

Let us consider the structure of the system of equations (2)

$$\bar{x}A = 0.$$

Each equation $L_i : \bar{x}A^i = 0$, where A^i is the ith column of the matrix A, corresponds to a transition t_i. The equation contains terms for all incident places. At the same time, its coefficients are equal to the weights of edges, and terms for the input places have the minus sign and those for the output places have the plus sign.

Thus, system (2) can be represented in the form

$$L = L_1 \wedge L_2 \wedge \ldots \wedge L_n. \tag{3}$$

THEOREM 2. An invariant $\bar{x}'$ of a Petri net N is an invariant of any of its functional subnets.

Proof. Since $\bar{x}'$ is an invariant of a Petri net N, $\bar{x}'$ is a nonnegative integer solution of system (3) and, hence, $\bar{x}'$ is a nonnegative integer solution of each equation L_i. Thus, $\bar{x}'$ is the solution of an arbitrary subset of the set $\{L_i\}$.

According to Statement 2 [10], a functional subnet Z', $Z' \succ N$, is generated by the set of its transitions T'. Hence, the equation corresponding to a transition of the subnet has the same form L_i as for the entire net since the subnet contains all the incident places of the initial net.

Thus, the system for finding the invariants of a functional subnet Z', $Z' \succ N$, is some subset of the set $\{L_i\}$ and the vector $\bar{x}'$ is its solution. Hence, $\bar{x}'$ is an invariant of the functional subnet Z'. The arbitrariness of the choice of a subnet $Z' \succ N$ in the above reasoning proves the theorem.

COROLLARY 2. All functional subnets of an invariant Petri net are invariant.

THEOREM 3. A Petri net N is invariant if and only if all its minimal functional subnets are invariant and there exists a common nonzero invariant of contact places.

Proof. In what follows, equivalent transformations of systems of equations are used lest the necessary and sufficient conditions should be independently proved. According to [10], the set of minimal functional subnets $\mathfrak{F} = \{Z^j\}$, $Z^j \succ N$, of an arbitrary Petri net N determines a partition of the set T into disjoint subsets T^j. Let the number of minimal functional subnets be equal to k. As is noted in the proof of Theorem 2, the equations contain terms for all the incident places. Thus, we have

$$L \Leftrightarrow L^1 \wedge L^2 \wedge \ldots \wedge L^k, \tag{4}$$

where L^j is the subsystem for a minimal functional subnet Z^j, $Z^j \succ N$. We note that if L^j has no solution, then L also has no solution (except for the trivial one).

Let G^j be the matrix of basic solutions of a subsystem L^j. Then the general solution of the subsystem L^j can be represented in the form

$$\bar{x} = \bar{u}^{\,j} G^j, \tag{5}$$

where $\bar{u}^{\,j}$ is an arbitrary vector of nonnegative integer numbers. According to (4), we have

$$L \Leftrightarrow \bar{x} = \bar{u}^{\,1} G^1 = \bar{u}^{\,2} G^2 = \ldots = \bar{u}^{\,k} G^k.$$

Hence, the system

$$\bar{x} = \bar{u}^{\,1} G^1 = \bar{u}^{\,2} G^2 = \ldots = \bar{u}^{\,k} G^k \tag{6}$$

is equivalent to the initial system of equations (2). Next, we will show that the solution of system (6) requires the consideration of a considerably smaller number of equations. Let us consider the following set of places of a Petri net N with the set of minimal functional subnets $\{Z^j \mid Z^j \succ N\}$:

$$P = Q^1 \cup Q^2 \cup \ldots \cup Q^k \cup C.$$

Here, Q^j is the set of internal places of a subnet Z^j and C is the set of its contact places. By definition, each internal place $p \in Q^j$ is incident only to transitions from the set T^j. Therefore, x_p corresponding to this place belongs to only the subsystem L^j. Hence, it is necessary to solve the equations for the contact places from the set C.

We will construct equations for the contact places of a net $p \in C$ since only these places are incident to more than one subnet. By Theorem 1, each contact place $p \in C$ is incident to no more than two functional subnets. Thus, we have the following equations:

$$\bar{u}^{\,j} G_p^j = \bar{u}^{\,l} G_p^l, \tag{7}$$

where j and l are the numbers of minimal functional subnets incident to a contact place $p \in C$ and G_p^j is the column corresponding to the place p in the matrix G^j. Equation (7) can be represented in the form

$$\bar{u}^{\,j} G_p^j - \bar{u}^{\,l} G_p^l = 0.$$

Thus, the system

$$\begin{cases} x_p = \bar{u}^{\,j} G_p^j, & p \in Q^j \vee p \in C, \\ \bar{u}^{\,j} G_p^j - \bar{u}^{\,l} G_p^l = 0, & p \in C, \end{cases} \tag{8}$$

is equivalent to the initial system (2), which proves the theorem.

Note that, in both cases considered in the proof, it is necessary, according to system (8), to solve a linear homogeneous system of equations.

COROLLARY 3. To compute the invariants of a Petri net, we should compute the invariants of its minimal functional subnets and then find the common invariants of contact places.

COROLLARY 4. Theorem 3 is also true for an arbitrary subset of functional subnets that determines a partition of the set of transitions of a Petri net.

COMPOSITION OF INVARIANTS OF FUNCTIONAL SUBNETS

Taking into account the results obtained in the previous section, we can formulate the compositional method of computation of invariants of a Petri net.

Stage 1. Decompose the Petri net into functional subnets.

Stage 2. Compute the invariants of each functional subnet, i.e., construct general solutions of homogeneous systems of equations (5).

Stage 3. Construct their composition, i.e., find the joint solution of system (8) for the set of contact places.

Stages 2 and 3 consist of solution of systems of linear homogeneous Diophantine equations in nonnegative integers. It is necessary to find the general solution of a system in the form of a linear combination of basic solutions. To this end, the methods described in [6–8] can be used.

Based on system (8), we write the following equations for contact places:

$$\bar{u}^{\,j} G_i^j - \bar{u}^{\,l} G_i^l = 0,$$

or, in the matrix form, we have

$$\| \bar{u}^{\,j} \;\; \bar{u}^{\,l} \| \begin{Vmatrix} G_i^j \\ -G_i^l \end{Vmatrix} = 0.$$

We number all the variables $\bar{u}^{\,j}$ so that the following general vector is obtained:

$$\bar{u} = \| \bar{u}^{\,1} \;\; \bar{u}^{\,2} \;\; \ldots \;\; \bar{u}^{\,k} \|,$$

and combine the matrices G_i^j and $-G_i^l$ into the common matrix K. Then we obtain the system

$$\bar{u} K = 0.$$

The system obtained is of the form (2) and, hence, its general solution is of the form (5)

$$\bar{u} = \bar{v} J. \qquad (9)$$

We will construct the combined matrix G of solutions (5) of system (2) for all functional subnets in such a manner that we have

$$\bar{x} = \bar{u} G. \qquad (10)$$

Substituting solution (9) in relation (10), we obtain

$$\bar{x} = \bar{v} J G.$$

Hence, we have

$$\bar{x} = \bar{v} H, \quad H = JG. \qquad (11)$$

Since only equivalent transformations has been used, the above reasoning proves the theorem presented below.

THEOREM 4. Relations (11) represent the general solution for the invariant of (2).

A drawback of well-known methods of solution of linear systems [7, 18] in nonnegative integers is their exponential computational complexity. For example, when the complexity equals $\sim 2^n$, where n is the number of nodes of a net, for finding the invariants of a model that consists of a hundred nodes, about 10^{30} operations should be executed at the worst. If we use a processor whose operating speed equals $\sim 10^{10}$ operations per second, then the computations will take more than 10^{12} years.

We estimate the overall acceleration of computations during the compositional computation of invariants. Let r be the maximum number of contact places or internal places of subnets. We represent $n = cr$, where c is some nonnegative constant. Then the complexity of computation of invariants with the help of decomposition can be estimated as $\sim 2^r$ since the complexity of decomposition, according to [10], is polynomial.

We represent the acceleration of computations in the form

$$2^n / 2^r = 2^{rc} / 2^r = 2^{r(c-1)} = 2^{n-r}. \qquad (12)$$

Thus, the obtained acceleration of computations is exponential.

COMPOSITION OF FUNDAMENTAL EQUATIONS OF FUNCTIONAL SUBNETS

For fundamental equations of Petri nets, the composition theorem that is given below and is similar to Theorem 3 for invariants is true; the process of its proof does not contain important distinctions.

THEOREM 5. The fundamental equation of a Petri net is solvable if and only if it is solvable for each minimal functional subnet and, moreover, there exists a joint solution for contact places.

The compositional method of solution of a fundamental equation can also be represented by the three stages considered in the preceding section. The difference is that nonhomogeneous systems of equations should be solved at Stages 2 and 3.

We similarly transform nonhomogeneous system (1). The general solution of the system for each functional subnet is of the form

$$\bar{y}^{j} = \bar{y}'^{j} + \bar{u}^{j} G^{j}. \tag{13}$$

Here, $\bar{u}^{j} G^{j}$ is the general solution of the homogeneous system and $\bar{y}'^{j} \in Y'^{j}$, where Y'^{j} is the set of minimal individual solutions of the nonhomogeneous system of equations.

We represent equations for contact variables in the form

$$\bar{y}_i'^{j} + \bar{u}^{j} G_i^{j} = \bar{y}_i'^{l} + \bar{u}^{l} G_i^{l}$$

and then obtain

$$\bar{u}^{j} G_i^{j} - \bar{u}^{l} G_i^{l} = \bar{b}_i', \quad \bar{b}_i' = \bar{y}_i'^{l} - \bar{y}_i'^{j}$$

or, in the matrix form,

$$\bar{u}K = \bar{b}'.$$

According to solution (13), the general solution of this system can be represented in the form

$$\bar{u} = \bar{u}' + \bar{v}J.$$

Using the combined matrix *G*, we represent general solution (13) in the form

$$\bar{y} = \bar{y}' + \bar{u}G$$

or

$$\bar{y} = \bar{y}' + (\bar{u}' + \bar{v}J)G = \bar{y}' + \bar{u}'G + \bar{v}JG$$

and obtain

$$\bar{y} = \bar{y}'' + \bar{v}H, \quad \bar{y}'' = \bar{y}' + \bar{u}'G, \quad H = JG. \tag{14}$$

Since only equivalent transformations have been used, the following theorem is proved.

THEOREM 6. Relation (14) represents the general solution of fundamental equation (1).

Note that the exponential acceleration of computations represented by formula (12) also takes place in the case when systems of fundamental equations for functional subnets have more than one minimal solution. Let each minimal functional subnet have no more than *n* minimal solutions. Then, to find common solutions for contact places, n^2 systems should be solved, and the polynomial multiplier in comparative estimates of exponential functions can be omitted.

EXAMPLE OF COMPOSITIONAL ANALYSIS

Let us perform the compositional analysis of the Petri net that is depicted in Fig. 1 and is a simplified model of the internet protocol BGP [13]. The model describes the asymmetric interaction of two systems. The first system is represented by the places $p_1 - p_5$ and transitions $t_1 - t_6$, and the second system is represented by the places $p_6 - p_{10}$ and transitions $t_7 - t_{12}$. The places $p_{11} - p_{14}$ correspond to the communication subsystem and simulate the standard messages OPEN, OPENCONFIRM, and KEEPALIVE. The model represents only the procedures of establishing and supporting a connection and abstracts from data exchanges that correct routing tables. Data exchange is realized in the state ESTABLISHED with the help of standard messages UPDATE.

A decomposition of the model into functional subnets is given in Fig. 2. Note that four depicted functional subnets Z^1, Z^2, Z^3, and Z^4 that describe the presented partition of the initial model are not minimal. For example, the subnet Z^2 is the sum of two minimal subnets generated by the transitions t_3 and t_4. Issues of composition of functional subnets on the basis of minimal ones are investigated in [10].

With the help of the Toudic method [6, 8], we obtain the following basic invariants of the listed subnets:

$$Z^1 : (x_1, x_2, x_3, x_5, x_{11}, x_{12}) = (u_1^1, u_2^1) G^1, \quad G^1 = \begin{pmatrix} 1\,1\,1\,1\,0\,0 \\ 1\,0\,1\,1\,1\,1 \end{pmatrix},$$

$$Z^2 : (x_3, x_4, x_5, x_{13}, x_{14}) = (u_1^2, u_2^2, u_3^2) G^2, \quad G^2 = \begin{pmatrix} 1\,1\,1\,0\,0 \\ 0\,0\,1\,0\,1 \\ 1\,0\,0\,1\,0 \end{pmatrix},$$

$$Z^3 : (x_6, x_7, x_8, x_{10}, x_{11}, x_{12}) = (u_1^3, u_2^3) G^3, \quad G^3 = \begin{pmatrix} 1\,1\,1\,1\,0\,0 \\ 0\,1\,0\,0\,1\,1 \end{pmatrix},$$

$$Z^4 : (x_8, x_9, x_{10}, x_{13}, x_{14}) = (u_1^4, u_2^4, u_3^4, u_4^4) G^4, \quad G^4 = \begin{pmatrix} 1\,1\,1\,0\,0 \\ 1\,1\,0\,0\,1 \\ 0\,1\,1\,1\,0 \\ 0\,1\,0\,1\,1 \end{pmatrix}.$$

According to the net of functional subnets of the model of the BGP protocol that are presented in Fig. 3, their composition is determined as a result of combination of eight contact places presented in the figure. We construct the following system of equations for contact places:

$$\begin{cases} p_3 : u_1^1 + u_2^1 - u_1^2 - u_3^2 = 0, \\ p_5 : u_1^1 + u_2^1 - u_1^2 - u_2^2 = 0, \\ p_8 : u_1^3 - u_1^4 - u_2^4 = 0, \\ p_{10} : u_1^3 - u_1^4 - u_3^4 = 0, \\ p_{11} : u_2^1 - u_2^3 = 0, \\ p_{12} : u_2^1 - u_2^3 = 0, \\ p_{13} : u_3^2 - u_3^4 - u_4^4 = 0, \\ p_{14} : u_2^2 - u_2^4 - u_4^4 = 0. \end{cases}$$

The basic solutions of the system are as follows:

$$J = \begin{pmatrix} 1\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0 \\ 0\;0\;0\;0\;0\;1\;0\;1\;0\;0\;0 \\ 0\;1\;1\;0\;0\;0\;1\;0\;0\;0\;0 \\ 1\;0\;0\;1\;1\;0\;0\;0\;0\;0\;1 \\ 0\;1\;0\;1\;1\;0\;1\;0\;0\;0\;1 \\ 1\;0\;0\;1\;1\;1\;0\;0\;1\;1\;0 \\ 0\;1\;0\;1\;1\;1\;1\;0\;1\;1\;0 \end{pmatrix}.$$

We will construct a combined matrix G from the matrices G^1, G^1, G^1, and G^1. The matrix G can be constructed by several methods that depend on the order of computation of invariants of contact places. Since each contact place is incident to two subnets, its invariant is computed by two methods and the matrix itself is of the form

$$G = \begin{pmatrix} 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0 \end{pmatrix}.$$

Multiplying the matrices, we obtain

$$H = JG = \begin{pmatrix} 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0 \\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 1 \\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1 \\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 2\ 1\ 0\ 0\ 1\ 1 \\ 1\ 0\ 1\ 0\ 1\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1 \end{pmatrix}.$$

The initial system has five basic solutions since the sixth solution is the sum of the second and fourth ones and the seventh solution is the sum of the second and fifth ones.

Thus, the model of the BGP protocol is invariant since, for example, all the components of the invariant

$$(\,2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 1\ 1\ 1\,)$$

that is the sum of the second, third, and fourth basic invariants are natural. Hence, the model of the protocol is safe and bounded.

We note that even such a relatively small model makes it possible to estimate the acceleration of computations. Instead of solution of a system of equations of order 12, five systems of an order that does not exceed 8 are solved. If the polynomial multipliers are not taken into consideration, then we obtain the sixteenfold ($2^{12} / 2^{8} = 16$) acceleration of computations.

CONCLUSIONS

In this article, the foundations of the compositional analysis of Petri nets are presented. The methodology described can be used in investigating properties of hierarchical models constructed from a collection of given subnets or properties of arbitrary given Petri nets considered as a composition of their minimal functional subnets.

Compositional analysis is destined for the acceleration of the analysis of properties of Petri nets with the help of methods based on fundamental equations of nets and on invariants. In investigating properties of Petri nets, systems of linear Diophantine equations must be solved in nonnegative integers. All well-known methods of solution of such systems have the exponential computational complexity.

The analysis of properties of models of real systems and processes consisting of thousand elements with the help of well-known earlier methods is a practically unsolvable problem since its solution requires computations for years. The use of compositional analysis makes it possible to obtain the exponential acceleration of computations and, hence, to substantially reduce problem times.

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